Note: the problems here do not necessarily cover all topics on the midterm; consult the study guide. An exam will be roughly 5 problems long, so there is more than one exam’s worth of questions here.

PROBLEMS

Problem 1. Show that, if \( k \geq 1 \) is an integer then the sequence

\[ a_n = 10^{-n^k} \]

converges superlinearly to zero but not of order \( \alpha \) for any \( \alpha > 1 \).

Problem 2. Show that the ‘practical’ version of Gauss-Seidel,

\[
x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)
\]

is the same as the matrix version,

\[
(L_A + D)x^{(k+1)} = -U_A x^{(k)} + b.
\]

Problem 3. a) Using Gaussian elimination, find the \( LU \) factorization of

\[
\begin{pmatrix}
2 & -2 & -4 \\
1 & 1 & -1 \\
3 & 7 & 5
\end{pmatrix}
\]

b) Do (a) but with partial pivoting (not scaled). Give \( L, U \) and the permutation matrix \( P \).

Problem 4. a) Suppose \( A, X, B \) are \( n \times n \) matrices and \( X \) and \( B \) have columns \( x_1, x_2, \ldots, x_n \) (all column vectors) and \( b_1, b_2, \ldots, b_n \). Show that solving the ‘matrix’ equation

\[ AX = B \]

is equivalent to solving \( Ax_i = b_i \) for \( i = 1, \ldots, n \).

b) Derive an algorithm for inverting an upper triangular (non-singular matrix). How many multiplications/divisions are required? \textit{Hint: Consider} \( UU^{-1} = I \).
**Problem 5.**  a) Given \(x^{(0)} = (0, 0)^T\), compute \(x^{(1)}\) and \(x^{(2)}\) (i.e. do two iterations) of the Jacobi and Gauss-Seidel method for the system \(Ax = b\) where
\[
A = \begin{bmatrix} 2 & 1 \\ 1/2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
Do you expect the iterations to converge?
b) Calculate the spectral radius for the iteration matrices in each case.

**Problem 6.** Do the following:
a) Show that if \(E\) is a matrix corresponding to swapping rows \(i\) and \(j\) then \(E^{-1} = E\) and \(E = E^T\).
b) Show that if \(P\) is a permutation matrix then \(P^{-1} = P^T\). *Hint: use (a).*
c) Give the permutation matrix \(P\) such that reverses the order of the rows for a 4 \(\times\) 4 matrix.

**Problem 7.** Suppose \(A \in \mathbb{R}^{n \times n}\) is invertible and \(b \in \mathbb{R}^n\) and \(k \geq 2\) is an integer. Give an efficient algorithm for computing the solution \(x\) to
\[
A^k x = b
\]
using \(2n^3/3 + O(kn^2)\) flops. Note that you may assume the Gaussian elimination algorithm is available for use and that it takes \(2n^3/3 + O(n^2)\) flops.

**Problem 8.** Suppose \(A\) is an invertible \(n \times n\) matrix with an LU factorization \(PA = LU\), where \(L\) was obtained using Gaussian elimination with partial pivoting (using exact arithmetic, so no numerical errors). Show that
\[
\|L\|_{\infty} \leq n.
\]

**Problem 9.** Prove the following miscellaneous facts:

a) If \(A\) is positive definite and \(B\) is invertible then \(BAB^T\) is positive definite.

b) If \(A \in \mathbb{R}^{n \times n}\) is invertible and \(||\cdot||\) is a vector norm on \(\mathbb{R}^n\) then
\[
||x||_A := ||Ax||
\]
defines a vector norm on \(\mathbb{R}^n\).

c) Prove that \(||x||_1 = |x_1| + \cdots + |x_n|\) (the 1-norm) is actually a norm.

d) Prove that \(||AB|| \leq ||A|| ||B||\) where \(||\cdot||\) is an induced matrix norm,
\[
||A|| = \max_{||x||=1} ||Ax||.
\]
Problem 10. Let $D = \text{diag}(d_1, d_2, \cdots, d_n)$ be a diagonal matrix.

a) What is $\kappa_2(D)$ (assuming that $D$ is invertible)?

b) Suppose $Dx = b$ and $D\tilde{x} = \tilde{b}$. When $D$ is an $n \times n$ diagonal matrix, derive the inequality
$$\frac{|x_i - \tilde{x}_i|}{|x_i|} \leq \frac{|b_i - \tilde{b}_i|}{|b_i|}, \quad \text{for } 1 \leq i \leq n.$$ 

How does this compare to the general bound from class (with $\kappa(D)$)?

Problem 11. Consider the function $f(x) = x^3 - x$ and Newton iterates $x_0, x_1, \cdots$.

a) What happens to the iterates if $x_0 > 1$? Explain your answer.

b) Find a value of $a$ such that if $|x_0| < a$ then $x_k$ converges to 0. Hint: write out the iteration explicitly.

c) Show that if $x_0 > 1/\sqrt{3}$ then $x_k$ converges to 1.

Problem 12. Suppose the Jacobi method is used to solve $Ax = b$ where

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$ 

a) Let $e^{(k)} = x^{(k)} - x$ be the error. Show that $e^{(k+1)} = Te^{(k)}$ (for a $T$ you should calculate) and that the components of the error oscillate without diverging or converging.

b) Find the iteration matrix for Gauss-Seidel. Does it converge?

Problem 13. Suppose the secant method is used to find a zero of $f(x) = x^2$. Derive the formula for the iterates $x_n$. Assuming that $0 < x_1 < x_0$ and

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = r,$$

find the value of $r$. Does this converge faster or slower than Newton’s method? Hint: write the iteration in terms of $r_n = x_{n+1}/x_n$ and $r_{n-1}$.

Problem 14. For this problem, you may use the fact that $\det(AB) = \det(A) \det(B)$ and that the determinant flips sign when rows of a matrix are swapped. That is, for the permutation matrix $P$ that swaps rows $i$ and $j$, we have $\det(P) = -1$.

Explain how to (numerically) compute the determinant of an invertible matrix $A \in \mathbb{R}^{n \times n}$ using $LU$ factorization.