Notes: There is no code to submit for this homework; you can use your code from the previous homework to do P6 (which requires some computation).

Problems

P1. Prove the following assertions:
   a) If \( f(x) \sim g(x) \) as \( x \to 0 \) then \( f = g + o(g) \).
   b) If \( a_n = O(1/n^p) \) as \( n \to \infty \) then \( a_n \) is also \( O(1/n^q) \) for all \( q < p \).
   c) Let \( f(x) = x \ln x \). Then \( f = O(x^p) \) for all \( p > 1 \) as \( x \to \infty \) but \( f \neq O(x) \).
   d) Let \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \). Then \( P = O(x^n) \) as \( x \to \infty \).

P2 (computing square roots). Suppose you are tasked with designing an algorithm to compute \( \sqrt{a} \) where \( a > 0 \) is a real number.

a) Show that Newton’s method applied to \( f(x) = x^2 - a \) will converge to \( \sqrt{a} \) if \( x_0 > 0 \).

b) Let \( e_k = x_k - \sqrt{a} \) be the error. Show, by directly manipulating the formula (not using any Taylor series approximations), that
\[
\frac{|e_{k+1}|}{\sqrt{a}} \leq \frac{1}{2} \left( \frac{|e_k|}{\sqrt{a}} \right)^2 \text{ if } x_k > \sqrt{a}.
\]
How many iterations are required to achieve a relative error of machine precision (\( \approx 2 \times 10^{-16} \)) if \( x_0 \) is chosen so that \( x_0 > \sqrt{a} \) and the initial error \( e_0 \) satisfies \( |e_0|/\sqrt{a} \leq 3/2 \)?

c) The algorithm can be improved by scaling so that the calculation is always done within a fixed interval. Let \( k \) be an integer such that
\[
b = 4^{-k}a \text{ satisfies } 1/4 \leq b \leq 1.
\]
We then use the method in part (a), but used to compute \( \sqrt{b} \). How many iterations are required to achieve machine precision in computing \( \sqrt{b} \)? Is any (relative) accuracy lost in going from \( \sqrt{b} \) to \( \sqrt{a} \) (the final result)?
P3. Suppose \( x^* \) is a root of \( f \) with multiplicity \( m > 1 \) and consider the iteration
\[
x_{k+1} = x_k - m \frac{f(x_k)}{f'(x_k)}.
\]
Assume that \( x_k \) converges to \( x^* \) and that \( f^{(m+1)}(x) \) is bounded. Show that \( e_{k+1} = O(|e_k|^2) \).

P4. Let \( f \) be a function with \( f'' > 0 \) and let \( x^* \) be a point such that \( f(x^*) = 0 \) with \( f'(x^*) < 0 \). The secant method is applied with starting values \( x_0 \) and \( x_1 \).

a) Show that if \( x_0, x_1 < x^* \) then \( x_2 < x^* \). Find analogous results for \( x_0, x_1 > x^* \) and when \( x_0, x_1 \) are on opposite sides of \( x^* \). You do not need to be rigorous if the claim is obvious from a sketch.

b) If \( x_1 < x^* < x_0 \), on which side of \( x^* \) is each element of the sequence \( x_k \)?

P5. Suppose a method is used to estimate a solution \( x^* \) to \( f(x^*) = 0 \) by generating a sequence of approximations \( \{x_k\} \) that converges to \( x^* \).

a) Suppose \( x^* = 0 \) and the result is that \( x_n = (0.99)^n \). How many iterations are required to have an error of less than \( 10^{-6} \)? If the iteration is stopped when \( |x_{n+1} - x_n| < 10^{-6} \), what is the absolute error?

b) In general, when is it true that \( |x_{n+1} - x_n| \) is a good estimate of the (absolute) error?

c) If the method is used to find the zero (at \( x^* = 1 \)) of \( f(x) = (x - 1)^{10} \) and stops when \( |f(x_n)| < \epsilon \), what is the error? In general, when is it true that the residual \( |f(x)| \) is a good estimate of the error? *Hint: Taylor expand \( f \) near \( x^* \).*

*Note: For (b) and (c), you only need to discuss briefly, rather than an exhaustive solution.*

P6 (computation). In calculating the modes of vibration of an oscillating object like a string, one may encounter an equation like
\[
\frac{1}{2} \lambda = \tan \lambda.
\]
The positive solutions \( 0 < \lambda_1 < \lambda_2 < \cdots \) are the fundamental frequencies.

Find the first five values of \( \lambda_k \) to six significant digits using the method of your choice. Briefly describe how you obtained the solutions. (*you don’t have to submit code; it just needs to be clear in your answer how the values were computed.*).