MATH 361S, SPRING 2018
HOMEWORK 3
DUE WEDNESDAY, FEB. 7

Theory Problems

Problem 1. Suppose $A$ is an invertible $n \times n$ matrix with an LU factorization $PA = LU$, where $L$ was obtained using Gaussian elimination with partial pivoting (using exact arithmetic, so no numerical errors). Show that $\|\tilde{L}\|_\infty \leq n$.

Problem 2. For the problems below, use the formal definitions:

$f(x) = O(g(x))$ as $x \to 0$ if there are constants $C, x_0$ such that $|f(x)| \leq C|g(x)|$ for $|x| < x_0$.

$f(x) = O(g(x))$ as $x \to \infty$ if there are constants $C, x_0$ such that $|f(x)| \leq C|g(x)|$ for $x > x_0$.

a) Prove that if $0 < m < n$ then $x^m = O(x^n)$ as $x \to \infty$.

b) Prove that if $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is a degree $n$ polynomial then $P_n = O(x^n)$ as $x \to \infty$.

c) Prove that if $f_1 = O(g)$ and $f_2 = O(h)$ then $f_1 f_2 = O(gh)$.

d) Under what conditions is $x^m = O(x^n)$ in the limit $x \to 0$? Prove your answer.

Problem 3. (K&C 4.4.2) Show that

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1, \quad x \in \mathbb{R}^n$$

and that equalities can occur (for non-zero vectors).

Problem 4. (K&C 4.4.11-12) Prove that

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|.$$ 

where $\|A\|_1$ is the subordinate matrix norm to the vector 1-norm. Use this norm to compute the condition number of

$$\begin{bmatrix} 1 & 1 + \epsilon \\ 1 - \epsilon & 1 \end{bmatrix}$$
i.e. find $\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1$. Hint: For the first part, you will need to interchange the order of a double sum.

Problem 5. Prove the following miscellaneous facts:

a) If $U$ is upper triangular and invertible then $U^{-1}$ is upper triangular.

b) If $U$ is unit upper triangular then $U^{-1}$ exists and is unit upper triangular.

c) If $A$ is positive definite and $B$ is invertible then $BAB^T$ is positive definite.

d) If $A \in \mathbb{R}^{n \times n}$ is invertible and $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$ then

$$\|x\|_A := \|Ax\|$$

defines a vector norm on $\mathbb{R}^n$.

Problem 6. A **diagonal matrix** $D$ is a matrix with all zeros except on the main diagonal. A square diagonal matrix has the form

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix},$$

which is written as $D = \text{diag}(d_1, d_2, \ldots d_n)$ for short.

a) What are $\det(D)$ and $\kappa(D)$?

b) Given a square matrix $A$, describe $DA$ and $AD$.

Problem 7. Let $\delta > 0$ be small, and suppose we want to solve $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & \delta & \delta \\ 1 & \delta & \delta \end{bmatrix}, \quad b = \begin{bmatrix} 2(1 + \delta) \\ -\delta \\ \delta \end{bmatrix}.$$

The exact solution $x = [\delta, -1, 1]^T$.

a) Let $D = \text{diag}(\delta^{1/2}, \delta^{-1/2}, \delta^{-1/2})$. Show (see note at the end) that $\kappa_\infty(A) \approx 2/\delta$ and $\kappa_\infty(DAD) \leq 4$.

b) Describe how to solve $Ax = b$ using the result in (a) while only solving well-conditioned linear systems (this is an example of **preconditioning**).

c) Let $\delta = 4 \times 10^{-8}$. Solve $Ax = b$ numerically to get an approximation $x_1$, then solve using the method in (b) to get another approximation $x_2$. Calculate the relative error in each case. You should find that the method in (b) is more reliable. **Note: whether this works or not may depend on the details of the computer arithmetic.**

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1Example borrowed from Golub and Van Loan, *Matrix Computations.*
**Suggestion:** You can use a computer algebra package to compute inverses for (a). Two options are MATLAB (consult the Mathworks page on symbolic expressions [here](#)) or Wolfram Alpha (easier) using the syntax

\[
\text{inverse of } \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 4 & 5 & 6 \end{bmatrix}
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