MATH 361, SPRING 2018
FINAL PROJECT INFORMATION

APRIL 4, 2018

DESCRIPTION

In class, we conduct a broad survey of topics as a way of illustrating fundamental numerical methods and key principles. In practice, things are not quite as structured. Problems that arise in applications can be messy and complicated. Computing solutions requires some experience and intuition to develop the right method and interpret the results.

The final project is an opportunity to explore a substantial mathematical problem in depth and the challenges involved in obtaining good numerical solutions. In doing so, you will have a chance to study extensions or new perspectives on the numerical methods we looked at in class, and see how they can be adapted to more substantial - and more realistic - problems.

1. REQUIREMENTS FOR THE FINAL PRODUCT

- **Research Paper**
  - About 15 pages, written in \LaTeX. (for completeness: length includes figures, code, references etc. where the text is double spaced, roughly 1-inch margins, although the length is really just a rough guideline.)
  - \LaTeX resources relevant to the project will be provided (but it will be beneficial to be acquainted with \LaTeX prior to any deadlines).
  - The paper should be written with the structure of a research article - abstract, introduction, discussion, numerical methods and results, analysis, and conclusions/summary.
  - There must be a significant component of numerical analysis in the topic, connecting to the themes of the course (error analysis, what makes the problem difficult to solve, how to address those problems, evaluation of methods, etc.)
  - There must also be a computational component: implement relevant algorithms and solve the problem in different ways, and (most importantly) discuss the results and use them as evidence for the numerical analysis component. Code can be written in MATLAB or python.

- **Presentation**
  - The presentation should be about 15 minutes long, with slides created in whatever way you like (suggestion: use beamer to create it in \LaTeX)
  - The main goal is to show the key ideas and insights gained from study of the problem. Time is limited, so you should attempt to distill your talk to the essence of the topic. Avoid technical details and connect to topics and themes from class when possible.
2. Schedule

Below are deadlines for intermediate and final products. All deadlines are 11:59PM on that day, submitted to Sakai. Later deadlines may be adjusted if necessary. Note: while it may be tempting to delay, it is important that you complete the requested work by each deadline to have adequate feedback.

Fri. 3/9: Topic proposal due. A ranked list of topics (at least three) either from the given list or topics discussed and approved by me.

Fri. 3/30: Abstract and list of sources due (written in \LaTeX). A tentative abstract outlining the planned content and scope of the project in the form of an abstract, and a (tentative) bibliography with annotations explaining why they are or will be useful.

Fri. 4/13: Draft of presentation slides due. The draft presentation should demonstrate what story you intend to tell, how you will frame things and what material will be presented.

Mon. 4/13: Rough draft or outline of the paper due. By this point the content and scope of the paper should be decided upon, i.e. you should have a clear plan on how to complete it. This should at least be a detailed outline (not all the content may be filled in or complete at this point).

Fri. 4/20: Final presentation due. The product should be finished by this point, although you may end up adjusting it afterwards before the actual schedule presentation date.

Tues. 5/1: Paper due (no extensions).

3. Details, content

3.1. Report. Some guidelines for the research paper:

General Structure: Model your report after a research article - it should introduce the problem of interest, provide necessary background and context, setup the problem to be studied in depth, the methods involved, the results, analysis thereof and concluding remarks (again, these do not all have to be separate sections named as such). Papers and articles you encounter in researching the topic should give you an indication of the expected style.

Exposition: For exposition, you should discuss the necessary background and central ideas of your topic, and summarize the important theoretical results. In doing so, try to synthesize what you have gathered from sources, emphasizing the key ideas. The intended audience should be students who have some familiarity with the fundamentals of numerical analysis (e.g. your peers in this class), so bear this in mind when deciding how to approach writing the report.

Numerical part: You will conduct some numerical experiments and analyze the results. While this is likely not original research, the actual code and data will be your own, and you may consider different parameter spaces or examples. There is room here for some creativity in what you choose to explore.

Code should be submitted as separate files. You may need to include a description of
the algorithm or snippets, but only do so if these are discussed in the text. Try to make plots and tables clear and concise, and make sure to include captions (use the figure or table environment and the caption command).

**Analysis:** Discuss some aspects of the topic, using your data as evidence or to illustrate key points (the nature of this will vary depending on the topic). This section should be your primary focus. Your discussion should connect back to the theory; the expository section is there to prepare the reader to understand the analysis. Common ‘numerical analysis’ ideas to address are

- What mathematical aspects of the problem create challenges for numerical methods? How are these issues addressed? (This is usually the fundamental question to ask)
- How does the error behave (rate, order of convergence; how long does it take before the error settles down into a predictable pattern)?
- What are the trade-offs involved in designing the method, e.g. efficiency vs. robustness and so on? How can the methods be made efficient/accurate (or other desirable properties)?
- What mathematics is involved in the derivation and analysis of the method (typically error analysis and stability)? Often there is some interesting linear algebra or real/complex analysis underlying the numerical problem.

3.2. **Presentation.** The purpose of the presentation is to share some of the key ideas and interesting features of your project with the class. Because of the time constraint, details on theory are impossible to present, so avoid them and focus on the important insights. Your computations and analysis should ideally be central to the presentation - these you can use to illustrate the main points (note: you may have to tailor the figures/data to be more presentation friendly).

It is probably best to decide upon a single central theme and organize the presentation around that theme; 15 minutes is really only enough time to explain one thing in a satisfactory way. As an example of budgeting time, you might spend five minutes introducing the topic, five minutes setting up the key problem and results, and five minutes illustrating it and interesting features.

Useful rule of thumbs: have about one slide per minute at most, often less; use slides as a presentation aid rather than a space for detailed text (emphasize figures and illustrative diagrams over equations).

Some aspects of the project will not end up in the presentation; only include what you can explain to the audience in the given time (you can, of course, allude to further details/insights, but do so carefully).

3.3. **Grading.** The project components will be weighted roughly as follows:

**Report:** 70% (see next section). A greater emphasis will be placed on the ‘numerical’ part.

**Presentation:** 30% (Slides and the presentation itself).
4. Suggested Topics

Below are some topics to choose from. Each has two components - a mathematical problem with some interesting numerical properties, and a relevant method or set of methods. In terms of focus, the balance of the two (the method itself vs. the application) depends on your interests. If you have an interest in a certain field, there may be room to adjust an existing topic to include it.

There is room for variations within each topic (there may be more than one method available to study for a given problem, or many problems to study for a given method).

You may suggest your own topic, subject to my approval, but try to choose something with the components listed above. A good topic should be able to address the goals enumerated in the previous sections.

Regardless of what you intend to select, you should discuss your thoughts with me at some point before topic proposals are due. Note that you will select a ranked list of topics, so it is possible you do not end up with your first choice.

Below are the provided topics. The area of the topic is labeled in blue and some ‘search keywords’ to consider when researching the topic are in purple. Note that you do not need much (if any) prior background for most of the topics beyond what will be covered in class.

**Minimization:** (Non-linear optimization; linear algebra) The conjugate gradient algorithm is a popular method for finding the minimum of a function. Minimization is hard in general, full of complications and uncooperative problems that must be addressed when computing solutions. There are many applications for this (possibilities: energy minimization for problems in physics; systems that arise in solving PDEs) (Search: Conjugate gradient, convex optimization, steepest descent, calculus of variations)

**Complex Newton’s method:** (Nonlinear equations, complex variables) How do we solve an equation \( f(z) = 0 \) for a complex function \( f \)? The convergence behavior can be surprisingly complicated (fractals lurk here). (Complex Newton’s method; Newton fractal)

**Quasi-Newton methods:** (Nonlinear systems) Often we need to solve equations \( F(x) = 0 \) where \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a complicated function. Quasi-Newton methods are a class of methods designed to overcome the limitations of regular Newton’s method, leading to more powerful and robust algorithms. There are a number of problems one could look at here. (Broyden’s method; Levenberg-Marquardt, DFP (Davidon-Fletcher-Powell))

**Graph connectivity:** (Eigenvalues, graph theory) Given a network of nodes, one is sometimes interested in determining how ‘connected’ the nodes are to each other. One can do so by calculating eigenvalues and eigenvectors - but the matrix involved can be quite large if the network is large. (Fiedler vector; algebraic connectivity; graph Laplacian)

**Markov chains:** (Eigenvalues, probability, graphs) Markov chains describe a system that moves between discrete states with certain probabilities. To determine how much time is spent in each state, one must calculate eigenvectors and eigenvalues. (Example: Google’s PageRank algorithm, which uses this idea to measure the popularity of a website by how
Orbital mechanics: (ODEs, physics) Simulate orbiting celestial bodies - perhaps including the infamous N-body problem, which exhibits intricate, unpredictable solutions (what does a planet do when orbiting two suns?). What are the numerical challenges involved in getting orbits that make sense? (Three-body problem, numerical methods for orbits/n-body problem)

Stiff and ill-behaved ODEs: (ODEs, stiffness) Chemical reactions can be described by systems of ODEs. Often, they involve a large number of reagents, and the resulting equations are stiff (a kind of numerical sensitivity that must be handled carefully). Note that there are many other applications where stiff ODEs arise, so ‘chemical reactions’ are easy to replace in this topic (e.g. the Fitzhugh-Nagumo model of neuron activity, solid mechanics, and much more.)

Chaotic systems: (ODEs) Chaotic systems are highly sensitive to small changes - making them notoriously difficult to solve numerically. Solving them, however, is rewarding as there is a remarkably rich structure. (Lorenz equations; chaotic dynamical systems)

Poisson’s equation: (Big linear systems, PDEs) One of the most ubiquitous equations in the sciences, Poisson’s equation describes everything from charge and heat distributions to stresses in solids and fluid flow. The discretized system leads to a large, sparse linear system that must be solved efficiently - and a number of methods are available. (Note: there are closely related equations with similar numerics as well).

Random walks: (ODEs, probability) How do we add randomness to ODEs? How are random processes simulated? (Euler-Maruyama method; Stochastic differential equations; random walks)

Delay differential equations: (ODEs) A delay differential equation depends on its past history - arising in many applications (electrical engineering, biology, population dynamics and more). This little detail makes even the simplest of DDEs challenging to solve and leads to interesting behavior.

Boundary value problems: (ODEs, nonlinear equations) Boundary value problems (ODEs with conditions imposed on a boundary) are much more difficult to solve than initial value problems - and often do not want to cooperate when being solved numerically. Two common approaches are shooting and finite difference methods, which have various advantages and disadvantages. (Applications: there are many - in physics and engineering and more, e.g. finding periodic solutions of an oscillator or the deformation of a solid)

Singular integrals: (Quadrature) Singular integrals are integrals of functions that have singularities. Such integrals appear frequently in models in electrostatics, fluid dynamics, swarming... The standard methods do not work if the integrand is not finite, so they have to be adjusted. Because they are often badly behaved numerically, computing solutions requires great care.

Condition number estimation: (Linear algebra) To get error estimates for problems involving $A^{-1}$, we would like to compute the condition number $\kappa(A)$. But doing so directly
requires $A^{-1}$ - which is a problem. A number of good methods to estimate $\kappa(A)$ have been developed, some probabilistic, some not, which would be interesting to study.

**Other topics:** Suggestions are welcome! Discuss potential topics with me and we can figure out something that works.

5. **Even more topics**

**Objects in a fluid:** (ODEs, Fluid dynamics) What happens when particles are dropped or mixed into a fluid? When vortices form (like the sort generated in the wake of an airplane or by a swimming jellyfish), how do they interact with the flow? As with the three-body problem in orbital mechanics, objects in a fluid interact to produce complicated and fascinating patterns. Their movements can be simulated using systems of ODEs which can be delicate to solve because of the instabilities involved. (Point vortices)