1. THE TOOLBOX: GEOMETRIC TOOLS

**Nullclines, direction field:** Solutions are horizontal/vertical along each nullcline. Between nullclines are regions $S$ where the vector field has a certain sign ($x', y'$ positive or negative). Solutions can only enter $S$ through certain faces (where the field points in) and can only exit through others (where it points out).

**Invariant sets (forward, barriers):** Nullclines can provide forward/backward barriers due to the sign of the vector field (inward/outward). All solutions are invariant sets and form barriers (solutions cannot cross).

Invariant sets give a quick way to ensure solutions stay in useful bounds (e.g. $x, y > 0$ to allow calculations that require $x$ and $y$ to be positive).

**Local behavior (linearized):** When the linearization is accurate (see theorem in next section), the phase portrait near the equilibrium looks like its linearized (LCC) phase plane. This is most useful for saddle points (for the slope of the stable/unstable manifolds as they enter/leave the equilibrium).

**Solutions must ‘go somewhere’:** A solution cannot ‘stop’ except at an equilibrium point; in this case it converges to the equilibrium in infinite time (does not reach it exactly). This plus the direction field and the above often are enough to deduce its path.\(^1\)

**Symmetries:** If there is a reflection symmetry across a line, then the phase portrait is the same across that line. If time is also reversed, the directions of the solutions are reversed. Often, the symmetry is $y \rightarrow -y$ or $x \rightarrow -x$, possibly with $t \rightarrow -t$. To check, change variables (e.g. replace $y$ with $-y$ and $t$ with $-t$) and see if the ODE system is the same.

Symmetry can be used to draw conclusions about stability, e.g. if an equilibrium with $y > 0$ is stable and there is a symmetry $y \rightarrow -y, t \rightarrow -t$ then its reflection is unstable.

---

\(^1\)In 1d, it was not hard for us to argue that solutions must increase/decrease along the phase line and nothing else. There is a corresponding result on planar systems that states the allowed behavior, the **Poincaré-Bendixson** theorem, which is beyond the scope of the informal discussion here.
2. THE TOOLBOX: ANALYSIS

**Linearization:** The linearized system at an equilibrium point is accurate if the eigenvalues have non-zero real part. Saddles/nodes/spirals are the good cases; centers and $\lambda = 0$ are the bad cases. In particular, this tells us the stability (asymptotically stable or unstable) of the equilibrium.

**Lyapunov’s theorems:** Given a Lyapunov function $E$, we can conclude that solutions must approach the set where $\dot{E} = 0$. This can be used to get ‘global’ information about convergence. In particular, if there is a unique minimum, solutions must approach it. In short: solutions want to flow towards regions of lower energy.

Unfortunately, there is no easy way to find a Lyapunov function (or to tell at a glance when one exists). Guessing $x^{2m} + y^{2n}$ (usually $x^2 + y^2$) or something similar can work.

**Conserved quantities:** When available, even better than Lyapunov functions. Use the $\dot{y}/\dot{x}$ trick to find one. Solution curves are contours of the conserved quantity $E$ (which is complete information for the phase portrait).

Conservative systems tend to have closed orbits (where the contours are closed curves). Around minima/maxima of $E$, solutions are closed and orbit around a center.

Other equilibria are usually saddle points (where $E$ has neither a minimum/maximum); linearization applies here to examine the local behavior.

3. PHASE PORTRAITS: PROCEDURES

A summary of the procedure; note that most steps can be re-ordered as needed.

1) **(geometry)** Identify the equilibria, nullclines and the (rough) direction field ($x', y'$ increasing or decreasing). Put directions on nullclines to indicate how solutions cross. Look for any easy invariant sets and symmetries (note that a symmetry cuts the analysis in half; just reflect to get the other half).

2) **(simple analysis)** Compute the linearization at each equilibrium. If it is valid, sketch in the local behavior from the linearized phase portrait.

3) **(theorems)** If available, use a conserved quantity and make the typical arguments (draw contours, etc.) to draw the phase portrait.

    Look for Lyapunov functions, and use them to conclude that solutions have to converge to certain sets. Where relevant, argue that approaching $\{\dot{E} = 0\}$ means convergence to an equilibrium (we won’t consider the other case in this course).

4) **(putting it together)** Sketch solution curves using all this information. In particular, draw the stable/unstable manifolds of any saddle points (they are often important!)

    **Note:** in some cases, (1)-(3) are not enough to know for sure what solutions do; in this case sketch a ‘plausible’ phase portrait. Harder things to argue: solutions are bounded vs. diverge; closed orbits or centers when not in the conservative case.

---

2For problems in this course, you will be prompted to do so when it is available.
4. Example: the pendulum (damped/undamped)

4.1. Pendulum (energy; phase portrait). Consider a pendulum with a mass at the end that swings back and forth with angular position $\theta(t)$ from the bottom (at rest, $\theta = 0$). When its swinging is wider, this 'linear' behavior is no longer valid. By some simple physics, it is not hard to derive the system

$$\dot{\theta} = v$$
$$\dot{v} = -\sin \theta,$$

where $v$ is the angular velocity (physical constants set to 1). Note that if $|\theta| \ll 1$ then

$$\dot{v} = -\sin \theta \approx -\theta$$

so it reduces to the 'linear oscillator' we saw previously for small oscillations ($\ddot{\theta} + \theta = 0$).

![Diagram of pendulum phase portrait]

Main goal: There are two types of behavior. For small total energy, it will have a 'swinging' motion that moves back and forth with amplitude $\theta_m$ (from $-\theta_m$ to $\theta_m$ and back). With enough energy, it can rotate over the maximum at $\theta = \pi$, and then rotates around instead.

By constructing the phase portrait, we can distinguish between initial states that lead to swinging vs. rotation.

Remark (periodicity): We consider the $(\theta, v)$ phase plane in its 'flat' form, where $\theta \in (-\infty, \infty)$. In reality, the $\theta$ direction has period $2\pi$; the phase portrait will repeat. It will be useful to consider the flat phase plane, even if there is some duplicate behavior.\(^3\)

To draw the phase portrait, we follow the usual steps (see Figure 2 for the diagram).

Setup: The preliminary calculations and setup required to draw the phase portrait.

Equilibria, linearization: The equilibria in the flat plane are at

$$A_n = (2n\pi, 0), \quad B_n = (-\pi + 2n\pi, 0), \quad n \in \mathbb{Z}$$

which correspond to the two true equilibria at $A = (0, 0)$ and $B = (\pi, 0)$. The linearization predicts a center for $A$ (to be addressed via a conserved quantity instead).

\(^3\)The most precise way to illustrate this effect is by plotting the phase plane on a cylinder, with the $\theta = 0$ and $\theta = 2\pi$ ends lining up.
Figure 1. Undamped pendulum; phase portrait. Red lines: stable/unstable manifolds of the saddle points.

For each $B_n$, we have

$$J = \begin{bmatrix} 0 & 1 \\ -\cos \theta & 0 \end{bmatrix} \implies J \big|_{B_n} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so it is a saddle point with eigenvalues/vectors (same for all the $B_n$’s)

$$\lambda = 1, \quad v = (1, 1)^T, \quad \lambda = -1, \quad v = (1, -1)^T.$$ 

The unstable manifold leaves $B$ with slope one; the stable manifold enters with slope $-1$.

**Symmetry:** The system has a reflection symmetry

$$v \to -v, \quad t \to -t$$

so solutions in the lower half of the phase plane are reflections of the upper half across $v = 0$ ($v \to -v$), with directions reversed ($t \to -t$).

**Energy:** The pendulum has a conserved quantity (in this case, total mechanical energy)

$$E(\theta, v) = \frac{1}{2}v^2 - \cos \theta.$$ \hspace{1cm} (1)

Note that the energy has a minimum at $A_n = (2n\pi, 0)$; the Hessian is

$$H = \begin{bmatrix} \cos \theta & 0 \\ 0 & 1 \end{bmatrix} \implies H \big|_{A_n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Analysis:** Let

$$\Gamma^+ = \text{unstable manifold of } B_0 \text{ in the } v > 0 \text{ half.}$$
This curve is a solution that leaves $B_0$ with slope 1 in the $+\theta$ direction, starting at $(-\pi, 0)$. It has energy (conserved) $E(-\pi, 0) = 1$, and so from (1), this curve is described by
\[ v = \sqrt{2 + 2 \cos \theta}. \]
Following it for increasing theta, we see that it continues up to $\theta = \pi$ where it hits $B_1(\pi, 0)$. But this is a saddle point, so it must be part of the stable manifold of $B_1$. Thus $\Gamma^+$ is an orbit from $B_0$ to $B_1$, which is called a heteroclinic orbit.\(^\text{4}\)

Similarly, let
\[ \Gamma^- = \text{unstable manifold of } B_1 \text{ in the } v < 0 \text{ half.} \]
This curve is given by $v = -\sqrt{2 + \cos \theta}$, and the same analysis shows that it connects $B_1$ to $B_0$ going in the $-\theta$ direction. This also follows directly from the symmetry.

Now define
\[ S = \text{region enclosed by } \Gamma^- , \Gamma^+. \]
This set is a union of blobs (closed oval-ish regions), each containing a center $A_n$. Using the energy, we see that this set is given explicitly by
\[ S = \{ (\theta, v) : E(\theta, v) < 1 \} \]
since the value of $E$ on the boundary is 1.

**Conclusions:** We can now draw the phase portrait and describe qualitative behavior (see the figure; we’ve justified all the features by this point).

- **Special orbits:** The heteroclinic orbits $\Gamma^\pm$ correspond to the exceptional case where the pendulum starts at the top ($\theta = \pi$). If pushed slightly, it will swing around and have just enough energy to come back to $\theta = \pi$ but not enough to go over.

- **Periodic orbits (swinging):** The equilibrium at $A = (0,0)$ is a center by the theorem for conservative systems. It is not hard to see that all solutions enclosed by $\Gamma_\pm$ are closed curves from (where the pendulum swings back and forth with a maximum angle $\theta_m < \pi$, i.e. from $-\theta_m$ to $\theta_m$.

  If the pendulum position/velocity start in $S$, it will swing back and forth.

- **Periodic orbits (rotating):** Outside of $S$, solutions have enough energy to go over the top and swing completely around (crossing $\theta = \pi$). On the flat phase plane, such solutions do not form closed orbits: instead, they just continue with increasing $\theta$ forever. Of course, in reality, they are periodic. Note that the $\Gamma^\pm$ curves form barriers that keep them away from the stable equilibria.

\(^\text{4}\)If we consider the $B_n$’s to be all the same equilibrium (with $\theta$ periodic) then this is a homoclinic orbit connecting $B$ to itself.
5. Damped pendulum

Now consider the damped pendulum
\[
\dot{\theta} = v \\
\dot{v} = -\sin \theta - \beta v
\]
where there is some damping force (like friction from air or at the attachment point) that slows it down; this is more realistic for your average swinging object. Assume
\[
\beta \ll 1
\]
(a small amount of damping, called ‘underdamped’).

**Main question:** In general, how does the presence of damping ($\beta > 0$) distort the phase portrait in the conserved case?

In particular, does the motion of the pendulum always slow to a stop? How many rotations occur (if any) before its motion settles down (from rotation to swinging)?

We can answer this question by showing the equilibria at $(2n\pi, 0)$ are asymptotically stable and identifying the region where solutions are attracted to each.

**Preliminaries:** As before, define the energy
\[
E(\theta, v) = \frac{1}{2} v^2 - \cos \theta.
\]
Notice that now it is not conserved:
\[
\dot{E} = v \dot{v} + v \sin \theta = -\beta v^2.
\]
It follows that
\[
E(\theta, v) \text{ is decreasing (except when } v = 0). \tag{2}
\]
This is a Lyapunov function for the minima at $A_n = (2n\pi, 0)$, so by the first theorem, each $A_n$ is asymptotically stable. It does not apply to the $B_n$’s as they are not minima of $E$.

**Linearization:**
\[
J = \begin{bmatrix} 0 & 1 \\ -\cos \theta & -\beta \end{bmatrix} \implies J|_{A} = \begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}, \quad J|_{B^\pm} = \begin{bmatrix} 0 & 1 \\ 1 & -\beta \end{bmatrix}
\]
The eigenvalues for $A$ are
\[
\lambda = \frac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4}) = -\frac{\beta}{2} \pm i\frac{\sqrt{4 - \beta^2}}{2}
\]
since $0 < \beta \ll 1$. It follows that $A$ is a stable spiral (consistent with the Lyapunov argument).

The eigenvalues for $B$ and eigenvectors are
\[
\lambda = \frac{1}{2}(-\beta \pm \sqrt{\beta^2 + 4}), \quad \mathbf{v} = (1, \lambda)^T
\]
which are both real, opposite signs so $B$ is a saddle point (same as before), but the eigenvectors are not quite of slope $\pm 1$. Note that if $\beta$ is small then $\lambda^- \approx -\beta$ and $\lambda^+ \approx 1 - \beta/2$.

**Further analysis:** The Lyapunov function provides much more information; this plus the phase portrait gives us the tools to answer the main question.

**Global stability:** The energy is decreasing everywhere except $\{v = 0\}$. By the second Lyapunov theorem, all solutions must tend towards this line. It is clear from the phase portrait that this can only happen if they approach one of the stable equilibria (otherwise they will rotate around). Thus, the theorem says that

$$\lim_{t \to \infty} (\theta(t), v(t)) = (2n\pi, 0)$$

for all solutions, $n \in \mathbb{Z}$.

Every solution eventually spirals into a minimum at $(2n\pi, 0)$. The value of $n$ is the number of full rotations that occur before it settles into a decaying swinging motion.

The stable manifolds $W_n$ of the $B_n$’s (see Figure) partition the plane into strips where solutions converge to each $A_n$; unlike the conserved case, these regions are not closed ovals due to the damping. For instance, if the starting $(\theta, v)$ is above $W_1$ and below $W_2$ then the pendulum will rotate around once and then settle down.

Comparing this to the conserved portrait, we see that when $\beta > 0$, the nice orbits connecting $B^{-}$ and $B^{+}$ no longer coincide: one gets sucked into $A$ and the other goes to $\infty$. 

---

**Figure 2.** Damped pendulum; phase portrait ($\beta = 0.25$). Red lines are stable manifolds of $B_0$ and $B_1$; magenta lines are stable manifolds ($W_1$ for $B_1$ etc.), computed numerically.
6. A NOTE: WHAT WE DON’T KNOW

Consider the Van der Pol oscillator

\[ x' = y, \quad y' = -x + y(1 - x^2). \]

There is a single equilibrium point at (0, 0). This ODE describes a non-linear damped oscillator where the damping is positive when \( x > 1 \) but negative when \( x < 1 \). When \( |x| \ll 1 \), we may expect it to behave like

\[ x'' - x' + x = 0 \tag{3} \]

which has negative damping: the system gains energy rather than decays. However, if \( x \) is large enough, the damping takes over and it wants to lose energy instead. Note that

\[ E(x, y) = x^2 + y^2 \implies \dot{E} = 2xy + 2y(-x + y(1 - x^2)) = 2y^2(1 - x^2) \]

so the energy arguments (at least for the physical energy) are not sufficient. It does suggest that if \( x \) is very large then \( \dot{E} \) is decreasing, i.e. the solution gets closer to (0, 0).

**Linearization:** There is one equilibrium point at (0, 0). The linearization gives

\[ J \bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad \lambda = \frac{1}{2} (1 \pm i\sqrt{3}) \implies \text{unstable spiral} \]

The type here is consistent with the observation that \( 1 - x^2 \approx 1 \) for small \( x \), leading to (3).

**Geometry:** The nullclines are \( y = 0 \) (for \( x' = 0 \)) and \( y = \frac{x}{1 - x^2} \) for \( y' = 0 \). We can sketch the nullclines, directions and spiral at (0, 0). However, it is not clear just from this what the phase portrait should be away from (0, 0) and \( \infty \).

A numerical plot reveals the true behavior. There is a unique periodic solution (called a limit cycle). Solutions that start inside this limit cycle spiral outward and ‘converge’ to it. This indicates that there is more to dynamics in 2d than we have addressed. For this, Strogatz’ *Non-linear dynamics and chaos* is a good introduction (or take Math 451, where such dynamics are a main focus).