Topics covered

• Solving eigenvalue problems with not-so-nice BCs
  ○ Physical example (cooling)
  ○ Graphical arguments (intersection)
  ○ Estimating eigenvalues
  ○ convergence rate; time to cool

1. Robin boundary conditions

1.1. Physical context: heating/cooling. Recall that the flux of heat for \( u_t = ku_{xx} \) is
\[
\text{flux} = -ku_x.
\]
Consider heat flow in an object of length \( L \) (e.g. a melting ice cube). If the ambient
temperature is \( T^\ast \), then heat flows out of the object according to Newton’s law of cooling,
\[
\begin{align*}
(x = 0 \text{ flux in}) &= -ku_x(0, t) = \beta(T^\ast - u(0, t)) \\
(x = L \text{ flux in}) &= ku_x(L, t) = \beta(T^\ast - u(L, t)).
\end{align*}
\]
where \( \beta > 0 \) is a constant. That is, the flux is proportional to the difference in temperature
and the flow is inward if the temperature in the object is higher. If \( \beta \) is small, then \( u_x \approx 0 \)
(nearly insulated); if \( \beta \) is large heat equilibrates quickly \( (u(0, t) \approx T^\ast) \).

Without loss of generality, \( T^\ast \) can be set to 0. Then this is a homogeneous BC with both \( u \)
and \( u_x \) - a Robin BC (sometimes called a ‘radiation BC’).

Of interest is knowing how fast the cooling occurs (the smallest \( \lambda \)) or if an equilibrium
exists at all (is \( \lambda > 0 \) for all eigenvalues?). The methods we have can be used to solve the
problem; the difference is that the eigenvalues cannot be found exactly.

1.2. The eigenvalue problem. Consider, as an example, \( \beta > 0 \) and
\[
-\phi'' = \lambda \phi, \quad \phi(0) = 0, \quad -\phi'(1) = \beta \phi(1).
\]
Solving the eigenvalue problem follows the standard process (you can replace Case 1 with a
Rayleigh quotient argument if \( \beta > 0 \)).
**Case I:** If $\lambda < 0$ then, with $\mu = \sqrt{-\lambda}$, the solution is
\[
\phi = c_1 e^{\mu x} + c_2 e^{-\mu x}.
\]
Applying the BCs at zero gives
\[
0 = \phi(0) = c_1 + c_2 \implies \phi = e^{\mu x} - e^{-\mu x}
\]
Now plug this into the BC at 1 to get
\[
-\phi'(1) = \beta \phi(1) \implies -\mu (e^{\mu} + e^{-\mu}) = \beta (e^{\mu} - e^{-\mu}).
\]
Rearranging, we have that an eigenvalue $\lambda = -\mu^2$ exists if $\mu > 0$ solves
\[
-\mu \frac{e^{2\mu} + 1}{e^{2\mu} - 1} = \beta. \tag{1.1}
\]
But $e^{2\mu} > 1$ if $\mu > 0$ so the LHS is negative (and $\beta > 0$ by assumption), so no solutions.

**Non-physical case:** If $\beta < 0$ then there might be a solution. Write
\[
 g(\mu) := \mu \frac{e^{2\mu} + 1}{e^{2\mu} - 1} = -\beta.
\]
A plot is shown below. A solution exists if $g(\mu)$ intersects the horizontal line $-\beta$.

There is either **no solution** or **exactly one solution**. To prove it, one can just show that
\[
g'(\mu) \geq 1 \text{ for } \mu > 0 \text{ and } g(0) = 1
\]
so there is a solution (and exactly one) only when $\beta < -1$.

**Case 2:** If $\lambda = 0$ then $\phi = c_1 x + c_2$. Applying the BCs,
\[
0 = \phi(0) \implies \phi = c_1 x, \quad \text{BC at 1} \implies -c_1 = \beta c_1
\]
so there is no solution unless $\beta = -1$ (since $c_1$ must be non-zero). Then
\[
\begin{cases}
\lambda = 0, \phi = x & \text{if } \beta = -1 \\
\lambda_0 \text{ not an e-value} & \text{if } \beta \neq -1
\end{cases}
\]
But $\beta > 0$ is assumed, so there is no zero eigenvalue.
**Case 3:** The most interesting case. We know from theory there should be an infinite sequence of solutions. Apply the \( x = 0 \) BC to get
\[
\phi = \sin \mu x, \quad \mu = \sqrt{\lambda}.
\]
Applying the BC at \( x = 1 \) then gives the equation for \( \mu \),
\[
-\mu \cos \mu = \beta \sin \mu.
\]
Thus (after rearranging), \( \lambda = \mu^2 \) is an eigenvalue if \( \mu > 0 \) solves
\[
-\frac{1}{\beta} \mu = \tan \mu. \tag{1.2}
\]
To find solutions, a graphical method is used: make sure the LHS and RHS are simple enough function and then plot them to find intersections. This is shown below with \(-\mu/\beta\) and \(\tan \mu\) plotted for \(\beta = 4\):

![Graphical method](image)

Note that \( \tan \mu \) has period \( \pi \). Then, to show that the eigenvalues exist:
- Observe that \( \tan \mu \) goes from \(-\infty \rightarrow \infty\) in \( I_n := [-\pi/2, \pi/2] + n\pi \).
- Thus the line must intersect \( \tan \mu \) once in each \( I_n \), starting with \( n = 1 \).
- There is no intersection in \([0, \pi/2]\) since the line is negative and \( \tan \mu \) is positive.
- It follows that the eigenvalues/functions are
\[
\lambda_n = \mu_n^2, \quad \phi_n = \sin \sqrt{\lambda_n} x \text{ for } n \geq 1 \tag{1.3}
\]
where \( \mu_n \) is the \( n \)-th positive solution to (1.2).

Moreover, we can locate the eigenvalues (approximately). By the intersection argument,
\[
-\pi/2 + n\pi < \mu_n < \pi/2 + n\pi
\]
which gives some information about the size of the eigenvalue, e.g.
\[
(\pi/2)^2 < \lambda_1 < (3\pi/2)^2.
\]
For example, for \( \beta = 4 \), the first few values (computed numerically from the plot) are
\[
\lambda_1 \approx 6.6, \quad \lambda_2 \approx 28.7, \quad \lambda_3 \approx 68.9.
\]
1.3. **Solving the PDE.** Now consider the IBVP for cooling of an object with length $L = 1$ with initial temperature $T_0 = 10^\circ$ (outside temperature $0^\circ$) and ‘slow’ leakage of heat at $x = 1$ (and ‘fast’ leakage at $x = 0$ so it stays at the external temperature):

$$
\begin{align*}
    u_t &= u_{xx}, \quad x \in [0,1], \quad t > 0 \\
    u(0,t) &= 0, \quad -u_x(1,t) = 4u(1,t) \\
    u(x,0) &= T_0
\end{align*}
$$

(1.4)

The goal: estimate the temperature as it cools completely, i.e. to show $u(x,t) \sim Ce^{-at}$ as $t \to \infty$ and to estimate the time $t^*$ required for the maximum temperature to be at most, say, $1^\circ$.

**Part I (eigenfunctions):** The eigenvalue problem for $\phi(x)$ (plug into the BCs) is

$$
\begin{align*}
    -\phi'' &= \lambda \phi, \quad \phi(0) = 0, \quad -\phi'(1) = 4\phi(1).
\end{align*}
$$

This is the example eigenvalue problem, so from (1.3) the eigenvalues/functions are

$$
\begin{align*}
    \lambda_n &= \mu_n^2, \quad \phi_n = \sin \sqrt{\lambda_n}x, \quad n \geq 1, \\
    \mu_n &= \text{zero in } [-\pi/2, \pi/2 + n\pi] \text{ of } -\frac{1}{4}\mu = \tan \mu.
\end{align*}
$$

(1.5)

By the theory, the eigenfunctions are orthogonal in the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$:

$$
\int_0^1 \sin(\sqrt{\lambda_m}x) \sin(\sqrt{\lambda_n}x) \, dx = \begin{cases}
    k_n & \text{if } m = n \\
    0 & \text{if } m \neq n
\end{cases}
$$

where $k_n = \langle \phi_n, \phi_n \rangle$ is some value that can be computed if needed:

$$
k_n = \int_0^1 \sin^2(\sqrt{\lambda_n}x) \, dx.
$$

Moreover, $\{\phi_n\}$ is an **orthogonal basis** for functions in $[0,1]$ by the theory. This means that $\phi_n$ has all the needed properties (orthogonal basis of eigenfunctions).

**Part II (PDE):** Write the initial condition $f(x) = T_0$ in terms of the basis,

$$
T_0 = \sum_{n \geq 1} f_n \phi_n, \quad f_n = \frac{\langle T_0, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{T_0}{k_n} \int_0^1 \sin \sqrt{\lambda_n}x \, dx.
$$

(1.6)

and write the solution as

$$
u = \sum_{n \geq 1} c_n(t) \phi_n(x)
$$

Now follow the usual process - the fact that the eigenfunctions are more complicated does not matter, since we have all the right properties. Plug the series into the PDE to get

$$
\begin{align*}
    \sum_{n \geq 1} c_n'(t) \phi_n(x) &= \sum_{n \geq 1} c_n(t) \phi''_n(x) \\
    \sum_{n \geq 1} c_n'(t) \phi_n(x) &= -\sum_{n \geq 1} \lambda_n c_n(t) \phi_n(x).
\end{align*}
$$

Similarly, plug into the IC to get $c_n(0) = f_n$. The IVP to solve for $c_n$ is then

$$
c_n'(t) + \lambda_n c_n(t) = 0, \quad c_n(0) = f_n.
$$

(1.7)
Since $\lambda_n > 0$, there is no zero eigenvalue to worry about and

$$c_n(t) = f_n e^{-\lambda_n t}, \quad n \geq 1.$$

Thus the solution is

$$u(x, t) = \sum_{n \geq 1} f_n e^{-\lambda_n t} \phi_n(x)$$

with $\lambda_n, \phi_n$ given by (1.5) and $f_n$ by (1.6).

**Cooling time (with numbers):** To answer the question of interest, it follows that

$$u(x, t) \sim f_1 e^{-\lambda_1 t} \phi_1(x) \text{ as } t \to \infty.$$  

Calculating $\lambda_1$ numerically (see previous section) we get $\lambda_1 \approx 6.6$. This gives

$$f_1 = \frac{\int_0^1 T_0 \sin \sqrt{\lambda_1} x \, dx}{\int_0^1 \sin^2 \sqrt{\lambda_1} x \, dx} \approx T_0 \frac{0.716}{0.589} \approx 1.22 T_0.$$  

using numerical integration. Now to get the time $t^*$ where the max value is at most $1^\circ$,

$$1 = \max_{x \in [0,1]} |u(x, t)| \leq f_1 e^{-\lambda_1 t^*}$$

$$\implies t^* = \frac{1}{\lambda_1} \ln(f_1) \approx \frac{1}{6.6} \ln(1.22 \cdot 10) = 0.379$$

in whatever units are appropriate.