Topics covered

• Periodic BCs: diffusion in a ring
  ○ Use of eigenfunctions (with a source term)
  ○ Eigenfunction basis with periodic BCs (Fourier series)
• Laplace’s equation in a disk
  ○ Solution (separation of variables)
  ○ Review: Cauchy-Euler equations

1. Periodic BCs: diffusion

Let’s consider a thin ring of radius 1/2 with temperature \( u(\theta, t) \) and a time-independent heat source. Picking functions for the IC/source, consider

\[
\begin{align*}
  u_t &= 4u_{\theta\theta} + \cos 2\theta, \quad \theta \in [0, 2\pi], \quad t > 0 \\
  u(\theta, 0) &= 1 + \sin 3\theta 
\end{align*}
\]

(1.1)

which can be derived from the Laplacian in polar coordinates (calculus exercise),

\[
  u_{xx} + u_{yy} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.
\]

Note that no boundary conditions are given for \( \theta \) - there are no ‘endpoints’. However, \( u(\theta, t) \) is 2\( \pi \)-periodic in \( \theta \).

We need two boundary conditions for the heat equation, so the above gives periodic BCs

\[
\begin{align*}
  u(0, t) &= u(2\pi, t), \quad u_\theta(0, t) = u_\theta(2\pi, t). 
\end{align*}
\]

(1.2)

The equation can be solved using a steady state, but we use the eigenfunction method directly to see how it works (this would be required if e.g. the heating changed over time like \( u_t = u_{\theta\theta} + e^{-t}g(\theta) \).

Part I (eigenfunctions): To solve, we first need to identify the eigenfunction basis. We can either recognize that \( Lu = -u_{\theta\theta} \) is the right choice, with

\[
  u_t = -4Lu + g(\theta)
\]

or use separation of variables. To use SoV, drop the source term and plug \( u = T(t)\phi(\theta) \) in to get (keeping the \( R^2 \) on the \( t \) side, which is easiest but not necessary)

\[
  \text{PDE} \implies \frac{1}{4} \frac{T'(t)}{T(t)} = \frac{\phi''(\theta)}{\phi(\theta)} = -\lambda
\]

BCs (1.2) \implies \( T(t)\phi(0) = T(t)\phi(2\pi), \quad T(t)\phi'(0) = T(t)\phi'(2\pi) \).
It follows that the eigenvalue problem for $\phi$ is

$$-\phi'' = \lambda \phi, \quad \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi)$$

so the operator is indeed $Lu = -u_{\theta\theta}$. We solved this problem in a homework [see HW 7]; the solution is the **Fourier series**

$$\lambda_0 = 0, \quad \phi_0 = \frac{1}{2} \quad \text{and} \quad \lambda_n = n^2, \quad \phi_n = \cos n\theta, \quad \psi_n = \sin n\theta, \quad n \geq 1.$$  

Each non-zero eigenvalue has two eigenfunctions (i.e. $a_n \cos n\theta + b_n \sin n\theta$ is an eigenfunction for $\lambda_n$ for any $a_n, b_n$).

**Part II (solve the problem):** Write the solution/source in the eigenfunction basis:

$$u(\theta,t) = a_0(t)\phi_0 + \sum_{n=1}^{\infty}(a_n(t)\phi_n + b_n(t)\psi_n)$$

for unknown coefficient functions $a_n, b_n$. To save time, we recognize the source term/IC are already in terms of the basis and only have a few terms:

$$\text{source} = \cos 2\theta = \phi_2, \quad \text{IC} = 1 + \sin 3\theta = 2\phi_0 + \psi_3.$$  

It follows that all terms are zero except for $n = 0, 2$ (cosine) and $n = 3$ (sine). To check carefully, either plug in the series (tedious) or note that the projected problems are

$$a'_n(t) = -\lambda_n a_n(t), \quad a_n(0) = 0 \quad \text{for} \quad n \neq 0, 2; \quad b'_n(t) = -\lambda_n b_n(t) = 0, \quad b_n(0) \quad \text{for} \quad n \neq 3.$$  

Now for the non-zero terms, project to get

$$a'_0(t) = 0, \quad a_0(0) = 2, \quad a'_2(t) = -4\lambda_2 a_2(t) + 1, \quad b'_3(t) = -4\lambda_3 b_3(t), \quad b_3(0) = 1.$$  

Now solve to get

$$a_0(t) = 2, \quad a_2 = \frac{1}{4\lambda_2}(1 - e^{-4\lambda_2 t}), \quad b_3 = e^{-4\lambda_3 t}.$$  

Thus

$$u(x,t) = 1 + \frac{1}{16}(1 - e^{-16t}) \cos 2\theta + e^{-36t} \sin 3\theta.$$  

Note that the steady state is visible here - the terms leftover when $t \to \infty$:

$$\pi(\theta) = \lim_{t \to \infty} u(x,t) = 1 + \frac{1}{16} \cos 2\theta. \quad (1.3)$$

**Remark:** For a steady state, we would have found $\pi$ to be the solution to

$$4\pi'' = -\cos 2\theta, \quad \int_0^{2\pi} \pi(\theta) \, d\theta = 1.$$  

The second condition comes from conservation of mass (check this!).
2. Laplace’s equation in a disk

Separation of variables can be used in geometries other than an interval/rectangle. To do so, we need to have variables such that the boundaries are separated - only one variable varies on each (e.g. only $x$ and only $y$ for the rectangle).

The $(x, y)$ coordinates cannot be used for a disk, but polar coordinates work. Laplace’s equation for $u(r, \theta)$ in a disk with a prescribed value $f(\theta)$ on the boundary is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad r \in (0, R), \quad \theta \in [0, 2\pi]$$

$$u(R, \theta) = f(\theta), \quad \theta \in [0, 2\pi]$$

We also need periodic boundary conditions in $\theta$ and a boundedness condition:

$$u(r, 0) = u(r, 2\pi), \quad u_r(r, 0) = u_r(0, 2\pi) \quad (2.1)$$

$$u(r, \theta) \text{ is bounded for } r \in [0, R] \quad (2.2)$$

$$u(R, \theta) = f(\theta)$$

**Part I (eigenfunctions):** The correct way to write the problem in operator terms is

$$u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} Lu = 0, \quad Lu = -u_{\theta\theta}.$$  

This is not obvious! To ‘derive it’, we can use separation of variables. Look for solutions $u = g(r)h(\theta)$.

Substituting into the PDE we get

$$g''(r)h(\theta) + \frac{1}{r} g'(r)h(\theta) + \frac{1}{r^2} g(r)h''(\theta) = 0$$

$$\Rightarrow \frac{r^2 g''(r) + rg'(r)}{g(r)} = -\lambda \frac{h''(\theta)}{h(\theta)} \quad (2.3)$$

With the periodic boundary conditions (2.1), we get a familiar eigenvalue problem:

$$h''(\theta) + \lambda h(\theta) = 0, \quad h(0) = h(2\pi), \quad h'(0) = h'(2\pi)$$

$$\Rightarrow h_0 = a_0, \lambda_0 = 0, \quad h_n(\theta) = \cos n\theta \text{ and } \sin n\theta, \quad \lambda_n = n^2, \quad n \geq 1. \quad (2.4)$$

**Caution:** As a warning, if the PDE is not homogeneous, SoV stops being useful here. At this point, we take the eigenfunctions/values and use the eigenfunction method.
We now solve for $g_n$ from (2.3):
\[ r^2 g''_n(r) + r g'_n(r) - n^2 g_n(r) = 0. \]

The ODE is a Cauchy-Euler equation with roots $\pm n$ (see review below; section 3); the basis solutions are $r^n$ and $r^{-n}$ so the general solution is
\[
    g_n = \begin{cases} 
        c_n r^n + d_n r^{-n} & n \geq 1 \\
        c_0 + d_0 \ln r & n = 0 
    \end{cases}
\]

By the boundedness condition (2.2), $d_n = 0$ ($r^{-n}$ and $\ln r$ are not finite at $r = 0$) so
\[
    g_n = c_n r^n, \quad n \geq 1.
\]

The separated solutions are then $g_n \cos n\theta$ and $g_n \sin n\theta$, or (grouping by eigenvalue),
\[
    u_0 = \frac{a_0}{2}, \quad u_n = r^n (a_n \cos n\theta + b_n \sin n\theta), \quad n \geq 1
\]

for arbitrary constants $a_n$ and $b_n$ (note that $1/2$ chosen to match the Fourier series).

**Part II (continuing with SoV):** Since the PDE is homogeneous, the solution is a linear combination of the $u_n$'s. With $\phi_0 = 1/2$, $\phi_n = \cos n\theta$ for $n \geq 1$ and $\psi_n = \sin n\theta$,
\[
    u(r, \theta) = a_0 \phi_0 + \sum_{n=1}^{\infty} r^n (a_n \phi_n + b_n \psi_n). \quad (2.5)
\]

Now to get the constants, impose the boundary condition at $r = R$:
\[
    f(\theta) = u(R, \theta) = a_0 \phi_0 + \sum_{n=1}^{\infty} R^n (a_n \phi_n + b_n \psi_n)
\]

so by the usual calculation for the coefficients (with $\langle f, g \rangle = \int_0^{2\pi} f(\theta) g(\theta) d\theta$)
\[
    a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \begin{cases} 
        \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & \text{for } n \geq 1 \\
        \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta & \text{for } n = 0
    \end{cases},
\]
\[
    b_n = \frac{\langle f, \psi_n \rangle}{\langle \psi_n, \psi_n \rangle} = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta.
\]

Note that this is just the Fourier series, but using the interval $[0, 2\pi]$. Using the inner product ensures that we have all the constants etc. right (vs. quoting the Fourier series formula).

The process is now complete; the solution is the Fourier series (2.5) with coefficients (2.6).

**Remark:** If the problem were inhomogeneous, we would consider
\[
    u(r, \theta) = a_0(r) \phi_0 + \sum_{n=1}^{\infty} a_n(r) \phi_n + b_n(r) \psi_n
\]
then plug this into the PDE and proceed as in the eigenfunction method.
2.1. **Other shapes.** The same method can be used to solve Laplace’s equation (or other PDEs) in any domain where the boundaries are all ‘separable’, i.e. of the form \( \text{variable} = \text{const.} \)

e.g. \( r = R \) for the circle or \( x = 0, B \) and \( y = 0, A \) for the rectangle’s four sides. Otherwise, other techniques (beyond the scope of the course) must be used. This is required to get eigenfunctions in **one direction** that don’t depend on the other (e.g. just \( \phi(\theta) \), not \( \phi(\theta, r) \)).

In polar coordinates, this means that sections \( (\Theta_1 \leq \theta \leq \Theta_2) \) and annuli \( (R_1 \leq r \leq R_2) \) and are also allowable domains (sketched below).

**Boundary conditions (polar):** An annulus and section are slightly different from the circle - some implied boundary conditions become explicit ones.
- In a **section**, there are flat boundaries out of the origin at the \( \theta \) endpoints.
- In an **annulus**, there are two boundaries in \( r \) where BCs can be specified.

For a section, the ‘periodic BCs’ are replaced by actual BCs at the \( \theta \) endpoints. For an annulus, the ‘bounded’ condition is replaced by the inner boundary.

**Example 1 (semicircle):** consider the semi-circle (upper left figure)

\[
\begin{align*}
    u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0, \quad r \in (0, 2), \quad \theta \in [0, \pi/2] \\
    u(r, 0) &= u(r, \pi), \quad r \in (0, R) \\
    u(2, \theta) &= f(\theta), \quad \theta \in [0, \pi]
\end{align*}
\]

Look for separated solutions \( g(r)\phi(\theta) \), leading to

\[
-\phi'' = \lambda \phi, \quad \phi(0) = \phi(\pi) \text{ in } [0, \pi], \quad g'' + \frac{1}{r} g' - \frac{\lambda}{r^2} g = 0.
\]

Note there are actual BCs at \( \theta = 0, \pi \). The eigenvalue problem is familiar and has solutions

\[
\phi_n = \sin(n\theta), \quad \lambda_n = n^2, \quad n \geq 1.
\]

The \( r \) equation is the same as for the full circle except that \( \lambda \neq 0 \), yielding

\[
g_n(\theta) = c_n r^n + d_n r^{-n} \implies g_n = c_n r^n \text{ for } n \geq 1.
\]

There is no \( n = 0 \) case to be concerned about. Thus the solution is, by superposition,

\[
    u(r, \theta) = \sum_{n \geq 1} c_n r^n \sin n\theta.
\]

and the \( c_n \)'s are determined by the BC at \( r = 1, u(1, \theta) = f(\theta) \).
Example 2 (annulus): Consider the half-annulus with Neumann BCs,
\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0, \quad r \in (1, 2), \quad \theta \in [0, \pi] \]

(flat end): \( u_\theta(r, 0) = u_\theta(r, \pi) \quad r \in (1, 2) \)

(inner): \( u(1, \theta) = 0, \quad \theta \in [0, \pi] \)

(outer): \( u(2, \theta) = f(\theta), \quad \theta \in [0, \pi] \)

The SoV steps work the same as before. The eigenfunctions/values are
\[ \phi_n(\theta) = \cos n\theta, \quad \lambda_n = n^2, \quad n \geq 0. \]

However, \( \lambda = 0 \) is now an eigenvalue. The solution has the form
\[ u(r, \theta) = g_0(r)\phi_0(\theta) + \sum_{n \geq 1} (c_n r^n + d_n r^{-n}) \phi_n(\theta). \]

As before, we have
\[ g_n(r) = \begin{cases} c_n r^n + d_n r^{-n} & n \geq 1 \\ c_0 + d_0 \ln r & n = 0 \end{cases} \]

but neither term is infinite in \([1, 2]\) so they cannot be excluded. The BCs at \( r = 1 \) and \( r = 2 \) both must be applied to find the coefficients. Let \( \langle f, g \rangle = \int_0^\pi f(\theta)g(\theta) \, d\theta \).

First apply the (inner) BC at \( r = 1 \):
\[ 0 = u(1, \theta) \implies g_n(0) = 0 \]
\[ \implies d_n = -c_n \text{ for } n \geq 1, \quad c_0 = 0. \]

Then apply the (outer) BC at \( r = 2 \):
\[ f(\theta) = u(2, \theta) = d_0 \ln 2 \phi_0 + \sum_{n \geq 1} c_n (2^n - 2^{-n}) \phi_n(\theta) \]
\[ \implies c_n = \frac{1}{2^n - 2^{-n}} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \text{ for } n \geq 1 \quad (2.7) \]

and for \( n = 0 \) (recall that \( \phi_0 = 1 \) here)
\[ d_0 \ln 2 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{1}{\pi} \int_0^\pi f(\theta) \, d\theta. \quad (2.8) \]

In summary, with \( d_0 \) and \( c_n \) given by \((2.8)\) and \((2.7)\),
\[ u(r, \theta) = d_0 \ln r + \sum_{n \geq 1} c_n \left( r^n - \frac{1}{r^n} \right) \cos n\theta \]
2.2. **Eigenfunctions in** r? Suppose now that we are instead solving the heat equation with radial symmetry, i.e.

\[ u_t = \frac{1}{r} (ru_r)_r, \quad u(1, t) = 0 \]

in a disk of radius one. Separating with \( u = T(t)\phi(r) \), we get a similar problem for R:

\[ \frac{T'(t)}{T(t)} = \frac{1}{r} (r\phi')' = -\lambda. \]

However, now \( r \) is the eigenfunction direction, and we have an eigenvalue problem

\[ \phi'' + \frac{1}{r} \phi + \lambda \phi = 0, \quad \phi(1) = 0, \quad \phi \text{ bounded in } [0, 1]. \]

Note that unlike the Laplace example, there is a BC at \( r = 1 \) and no \( 1/r^2 \) factor on the last term. This is not a Cauchy-Euler equation that can be solved exactly, and in fact its bounded solutions are much more complicated.

Here, the relevant eigenfunctions are **Bessel functions** (which won’t be detailed here). The point here is that not all eigenvalue problems will yield nice eigenfunctions, and the appropriate eigenfunctions depend on the geometry.
3. Review: Cauchy-Euler Equations

A type of ODE that can be solved exactly (appearing, previously, as P3 of HW4). A **Cauchy-Euler** or **equidimensional** ODE of second order has the form

$$ax^2y'' + bxy' + cy = 0, \quad x > 0 \text{ or } x < 0.$$  \hspace{1cm} (3.1)

To solve, guess a 'trial solution' of the form $x$ to a power. Since

$$y(x) = x^\gamma \implies xy' = \gamma x^\gamma, \quad x^2y'' = \gamma(\gamma - 1)x^\gamma$$

we have that $x^\gamma$ is a solution if and only if

$$p(\gamma) = a\gamma(\gamma - 1) + b\gamma + c = 0$$ \hspace{1cm} (3.2)

where $p(\gamma)$ is the 'characteristic polynomial'. We need two linearly independent solutions to solve (3.1). If $\gamma_1 \neq \gamma_2$ are real, we are done (two solutions). Otherwise:

- If $\gamma$ is a repeated root, then multiply by a factor of $\ln x$ (i.e. solns $x^{\gamma}$ and $x^{\gamma}\ln x$).

- $\gamma = s + \omega i$ is complex, take real/imaginary parts to get two solutions:

  $$x^r = e^{(s + \omega i)\ln x} \implies x^s \cos(\omega \ln x), \quad x^s \sin(\omega \ln x).$$

- For negative $x$, replace with $|x|$.

In summary, the solution procedure is:

1) Plug in the trial solution $x^\gamma$ and find the characteristic polynomial $p(\gamma)$.

2) Calculate the roots $\gamma_1, \gamma_2$ of $p(\gamma)$

3) The solution depends on the roots (three cases):

   - roots $\gamma_1 \neq \gamma_2$, real $\implies y = c_1|x|^\gamma + c_2|x|^\gamma$
   - root $\gamma$ (repeated), $\implies y = c_1|x|^\gamma + c_2|x|^\gamma \ln |x|$
   - roots $\gamma = s \pm \omega i$ (complex) $\implies y = c_1|x|^s \cos(\omega \ln |x|) + c_2|x|^s \sin(\omega \ln |x|)$

**Remark:** The cases are 'like a LCC equation, but with $\ln x$ instead of $x$' and the characteristic polynomial is (3.2) instead of $a\gamma^2 + b\gamma + c$ (for LCC). In fact, one can convert the Cauchy-Euler equation into an LCC one by using $\ln x = t$. 