1. THE EIGENFUNCTION METHOD TO SOLVE PDEs

We are now ready to demonstrate how to use the components derived thus far to solve the heat equation. The procedure here for the heat equation will extend nicely to a variety of other problems. For now, consider an initial boundary value problem of the form
\[ u_t = -Lu + h(x,t), \quad x \in (a,b), \quad t > 0 \]
with homogeneous boundary conditions at \( a \) and \( b \),
\[ u(x,0) = f(x) \]  
(1.1)
We seek a solution in terms of the eigenfunction basis
\[ u(x,t) = \sum_n c_n(t)\phi_n(x) \]
by finding simple ODEs to solve for the coefficients \( c_n(t) \). This form of the solution is called an eigenfunction expansion for \( u \) (or ‘eigenfunction series’) and each term \( c_n\phi_n(x) \) is a mode (or ‘Fourier mode’ or ‘eigenmode’).

Part 1: find the eigenfunction basis. The first step is to compute the basis. The eigenfunctions we need are the solutions to the eigenvalue problem
\[ L\phi = \lambda\phi, \quad \phi(x) \text{ satisfies the BCs for } u. \]  
(1.2)
By the theorem in ??, there is a sequence of eigenfunctions \( \{\phi_n\} \) with eigenvalues \( \{\lambda_n\} \) that form an orthogonal basis for \( L^2[a,b] \) (i.e. one with all the required properties).
If possible, we compute solutions explicitly via the standard procedure. Note that the BCs imposed on the eigenvalue problem must be homogeneous.

Now at each fixed time \( t \), the function \( u(x,t) \) is a function of \( x \) defined on \([a,b] \). It follows that there are coefficients \( c_n(t) \) such that

\[
   u(x,t) = \sum_n c_n(t) \phi_n(x).
\]

For each \( t \), \( \{c_n(t)\} \) is the set of coefficients for expressing \( u(x,t) \) in terms of the basis \( \{\phi_n\} \).

**Part 2: get equations for the coefficients:** Our objective now is to reduce the problem to ODEs for coefficients \( c_n(t) \). The order of steps can be changed here.

**Step 2a (Write known functions in the basis):** We express every known function in the problem (PDE and ICs) in the eigenfunction basis using the orthogonality rule (??). In this case, there are two such functions, the source term and the initial condition:

\[
   f(x) = \sum_n f_n \phi_n(x), \quad f_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle},
\]

\[
   h(x,t) = \sum_{n=0}^{\infty} h_n(t) \phi_n(x), \quad h_n(t) = \frac{\langle h(x,t), \phi_n(x) \rangle}{\langle \phi_n, \phi_n \rangle}.
\]

Here \( \langle f, g \rangle = \int_a^b f(x)g(x) \, dx \) is the \( L^2 \) inner product in \( x \); often, it is not worth calculating the integrals explicitly (they can be a mess). Note that in (1.5), the \( t \)-variable is not part of the integral, e.g.

\[
   h(x,t) = tx \implies \langle h, \phi_n \rangle = t \int_a^b x \phi_n(x) \, dx.
\]

**Step 2b (Plug series into the PDE):** We now ‘plug the series in’ to the PDE - which expands the PDE in terms of the eigenfunction basis. Taking this one part at a time\(^1\),

\[
   u_t = \frac{\partial}{\partial t} \left( \sum_n c_n(t) \phi_n(x) \right)
   = \sum_n c'_n(t) \phi_n(x), \quad (t\text{-derivs. only act on coeffs.})
   
   Lu = \sum_n c_n(t) L\phi_n
   = \sum n \lambda_n c_n(t) \phi_n(x) \quad (\phi_n \text{ is an eigenfunction})
\]

Plugging this into the PDE, we get

\[
   u_t = -Lu + h(x,t)
   \implies \sum_n c'_n(t) \phi_n(x) = - \sum_n \lambda_n c_n(t) \phi_n(x) + \sum_n h_n(t) \phi_n(x).
\]

\(^1\)After getting used to the process, you can shortcut the work here and skip some steps. They tend to be the same for most problems, but you should be careful and recognize when the steps must be changed.
The eigenfunction property means that the coefficient of \( \phi_n \) in \( Lu \) depends only on \( c_n(t) \). That is, the \( n \)-th term in the series is independent of the others.

Now since \( \{\phi_n\} \) is a basis, the coefficients on the left and right sides must be equal term-by-term. Alternatively,

\[
\sum_n (c'_n(t) + \lambda_n c_n(t) - h_n(t)) \phi_n(x) = 0 = \sum_n 0 \cdot \phi_n
\]

and since zero must have a unique representation in the basis,

\[
c'_n(t) + \lambda_n c_n(t) = h_n(t) \quad \text{for all } n. \tag{1.6}
\]

The difficult part is now over.

**Step 2c (solve the ODE):** Next, solve the ODE (1.6) for each \( n \). Note that sometimes, this will involve some case work (as we saw with Fourier series).

The solution will have some arbitrary constants. At this point, the series \( \sum c_n(t) \phi_n(x) \) is a ‘general solution’ to the PDE with the given BCs with the initial condition not yet imposed.

**Step 2d (apply initial conditions):** This can be done earlier. We turn the ODE into an IVP with a unique solution by applying the initial conditions (any ‘leftover’ equations not yet used). As in the PDE, plug in the series:

\[
u(x,0) = f(x) \implies \sum_n c_n(0) \phi_n(x) = \sum f_n \phi_n(x)
\]

which gives initial conditions for the \( c_n \)'s:

\[
c_n(0) = f_n. \tag{1.7}
\]

Solving (1.6) with (1.7) yields unique solutions for \( c_n(t) \), and we are done.

**Step 3 (summarize the solution):** As the steps can be long, it is useful to make the solution clear. You should check that all functions and constants are well-defined by some formula. We have obtained a solution

\[
u(x,t) = \sum_n c_n(t) \phi_n(x)
\]

to the IBVP (1.1) where the \( c_n \)'s and \( \phi_n \)'s are given by

\[
c_n(t) = \cdots, \quad \phi_n = \cdots
\]

(replace with formula computed in the procedure) and \( f_n, h_n(t) \) are given by (1.4) and (1.5).

Note that it is often best to leave expressions in terms of explicit integrals e.g. like

\[
f_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{\pi} \int_0^\pi x^2 \sin nx \, dx
\]

when computing would not give any new insight. The inner product shorthand can be used, but its exact definition should be clear (the interval etc.).
1.1. Example 1: no source; Dirichlet BCs. The simplest case. We solve

\[ u_t = u_{xx}, \quad x \in (0, 1), \; t > 0 \]  
\[ u(0, t) = 0, \quad u(1, t) = 0, \]  
\[ u(x, 0) = f(x). \]  

(1.8a) (1.8b) (1.8c)

The eigenvalues/eigenfunctions are (as calculated in previous sections)

\[ \lambda_n = n^2 \pi^2, \quad \phi_n = \sin n\pi x, \quad n \geq 1. \]  

(1.9)

Assuming the solution exists, it can be written in the eigenfunction basis as

\[ u(x, t) = \sum_{n=0}^{\infty} c_n(t) \phi_n(x). \]

There is only one function to write in the eigenfunction basis (note that \( \int_0^1 \phi_n^2 dx = 1/2 \) here)

\[ f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x), \quad f_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = 2 \int_0^1 f(x) \sin n\pi x \, dx. \]  

(1.10)

Substitute into the PDE (1.8a) and use the fact that \( -\phi_n'' = \lambda_n \phi \) to obtain

\[ \sum_{n=1}^{\infty} \left( c_n'(t) + \lambda_n c_n(t) \right) \phi_n(x) = 0. \]

Doing the same for the initial condition, we get the IC for \( c_n \):

\[ u(x, 0) = f(x) \implies \sum_{n=1}^{\infty} c_n(0) \phi_n(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \implies c_n(0) = f_n. \]

Equating coefficients of each basis function (to zero), we find that

\[ c_n'(t) + \lambda_n c_n(t) = 0, \quad c_n(0) = f_n. \]  

(1.11)

We are now done. The solution to the IBVP (1.8) is

\[ u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin n\pi x \quad \text{with } b_n \text{ given by (1.13).} \]  

(1.12)

Alternatively, we could summarize by writing that the solution is

\[ u(x, t) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t} \phi_n(x) \]

with eigenfunctions/values \( \phi_n, \lambda_n \) given by (1.9) and \( f_n \) by (1.10).

**Long-time behavior:** Note that every term in the solution (1.12) has a negative exponential (since all the eigenvalues are positive). Furthermore, terms further down in the series decay much faster since \( \lambda_n \) grows quadratically with \( n \). It follows (informally) that

\[ \lim_{t \to \infty} u(x, t) = 0 \]
independent of the initial condition \( f(x) \) (which just affects the \( b_n \)'s and not the eigenvalues).

Moreover, by approximating \( u \) by its first term,

\[
u(x, t) \approx f_1 e^{-\lambda_1 t} \phi_1(x) \quad \Rightarrow \quad \text{decays to zero at least as fast as } f_1 e^{-\lambda_1 t}.
\]

Thus, the **smallest eigenvalue** of a non-zero term gives the convergence rate for the solution (which is exponential).

As an explicit example, suppose the initial condition is

\[
f(x) = x(1 - x).
\]

After some laborious integration by parts, we get

\[
b_n = \frac{2(1 - (-1)^n)}{\pi^3 n^3} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi^3 n^3} & n \text{ odd.} \end{cases}
\]

The first few terms of the solution are

\[
u(x, t) = \frac{2}{\pi^3} e^{-\pi^2 t} \sin \pi x + \frac{2}{27\pi^3} e^{-9\pi^2 t} \sin 3\pi x + \cdots
\]

Note that \( u(x, t) \) is missing a \( \phi_2 = \sin 2\pi x \) term; this does not affect the convergence rate.

However, suppose instead that \( f(x) = x - 1/2 \). Then we have

\[
b_1 = 2 \int_0^1 (x - 1/2) \sin \pi x \, dx = 0, \quad b_2 = 2 \int_0^1 (x - 1/2) \sin 2\pi x \, dx = -1/\pi.
\]

the \( n = 1 \) term vanishes, so

\[
u(x, t) \approx -\frac{1}{\pi} e^{-\lambda_2 \pi^2 t} \phi_2(x) + \cdots \quad \text{as } t \to \infty
\]

and the convergence rate is given by the second eigenvalue \( \lambda_2 = 4\pi^2 \) (faster convergence!).
1.2. **Example 2: no source, Neumann BCs.** A variation - similar to Dirichlet, but with a crucial difference due to the zero eigenvalue. Here we seek a solution \( u(x, t) \) to the IBVP
\[
\frac{du}{dt} = u_{xx}, \quad x \in (0, 1), \ t > 0
\]  
with boundary and initial conditions
\[
\begin{align*}
\frac{du}{dx}(0, t) &= 0, & u_x(1, t) &= 0, & u(x, 0) &= f(x) .
\end{align*}
\]  
The eigenvalues/eigenfunctions are (again, computed earlier)
\[
\lambda_n = n^2 \pi^2, \quad \phi_n = \cos n\pi x, \quad n = 0, 1, 2, \ldots
\]  
**Note that** \( \lambda_n = 0 \) is an eigenvalue, unlike the previous case. Regardless, the process is the same and we end up with a solution (check this!)
\[
\begin{align*}
u(x, t) &= \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x
\end{align*}
\]
for constants \( a_n \) determined by the initial condition:
\[
f(x) = \sum_{n=0}^{\infty} a_n \cos n\pi x \implies a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.
\]
The integrals here are different for \( n = 0 \) and \( n \neq 0 \), which is typical, so we should be careful:
\[
\begin{align*}
a_0 &= \frac{\int_0^1 f(x) \, dx}{\int_0^1 1 \, dx} = \int_0^1 f(x) \, dx, \\
a_n &= \frac{\int_0^1 f(x) \cos n\pi x \, dx}{\int_0^1 \cos^2 n\pi x \, dx} = 2 \int_0^1 f(x) \cos n\pi x \, dx, \quad n \geq 1.
\end{align*}
\]
**Long-time behavior:** The zero eigenvalue changes the \( t \to \infty \) limit. We have
\[
\begin{align*}
u(x, t) &= a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x = a_0 + a_1 e^{-\pi^2 t} \cos \pi x + \cdots
\end{align*}
\]
or more abstractly,
\[
u(x, t) = a_0 \phi_0(x) + a_1 e^{-\lambda_1 t} \phi_1(x) + \cdots
\]
As \( t \to \infty \), the solution will approach the term for the zero eigenvalue:
\[
\lim_{t \to \infty} u(x, t) = a_0 \phi_0(x) = \text{(constant; value = } \int_0^1 f(x) \, dx \text{)}
\]
and the convergence is at least as fast at \( e^{-\lambda_1 t} \) (exponential, rate \( \lambda_1 \)). The constant \( a_0 \) is the average value of the initial distribution \( f(x) \).

This result confirms the intuitive notion that if you put something that diffuses into a closed container (e.g. tea in water), then over time the concentration of material will even out until it is uniform. The total amount of material does not change.
2. INDEPENDENCE OF MODES

Beyond computation, the eigenfunction basis tells us key information about the solution. Consider again the problem

\[ u_t = -Lu + h(x,t) , \]

(hom. BCs at \( x = a \) and \( x = b \))

\[ u(x,0) = f(x). \]  

(IBVP)

We saw that the solution is an eigenfunction expansion

\[ u(x,t) = \sum_n c_n(t) \phi_n(x) = \sum_n u_n(x,t) \]

where \( u_n(x,t) = c_n(t) \phi_n(x) \) is the \( n \)-th term, sometimes called the \( n \)-th mode of the series.

Notice that the coefficient equations are independent of each other. That is, each mode evolves independently (they do not interact). We can say more using a projection.

**Projection map:** Given an orthogonal basis \( \{ \phi_n \} \) in the inner product \( \langle f,g \rangle = \int_a^b f(x)g(x) \, dx \), the projection onto the \( n \)-th mode is

\[ P_n u = \frac{\langle u, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n. \]

This operator extracts the \( n \)-th mode of an eigenfunction series, e.g.

\[ f = \sum_n f_n \phi_n \implies P_n f = f_n \phi_n. \]

The map (from functions to scalars)

\[ u \mapsto \frac{\langle u, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \]

gives the \( n \)-th coefficient.

As before, suppose

\[ f = \sum_n f_n \phi_n, \quad h_n = \sum_n h_n(t) \phi_n(x). \]

Take the projection \( P_n \) of the problem (IBVP) to get a ‘projected’ problem. For the \( u \) parts:

\[ P_n(u_t) = P_n \left( \sum_n c_n'(t) \phi_n(x) \right) = c_n'(t) \phi_n(x) \]

\[ P_n(Lu) = P_n \left( \sum_n \lambda_n c_n(t) \phi_n(x) \right) = \lambda_n c_n(t) \phi_n(x) \]

This gives

\[
\begin{cases}
  u_t = -Lu + h(x,t) \\
  u(x,0) = f(x)
\end{cases} \quad \implies \quad P_n \implies \begin{cases}
  c_n'(t) \phi_n(x) = -\lambda_n c_n(t) + h_n(t) \phi_n(x) \\
  c_n(0) \phi_n(x) = f_n \phi_n(x)
\end{cases}
\]

It follows that the \( n \)-th mode \( u_n(x,t) \) solves the **projected** problem
Figure 1. Decomposing a system into its independent modes: the \( n \)-th mode of the input affects only the \( n \)-th mode of the solution.

\[
\begin{align*}
\text{inputs} & \quad \implies \quad \text{system} & \quad \implies \quad \text{response} \\
\text{IC source} & \quad \implies \quad \begin{cases} u_t = u_{xx} + h \\ u(x,0) = f \end{cases} & \quad \implies \quad \sum_n c_n(t)\phi_n \\
\text{system}_n & \\
\text{inputs} & \quad \implies \quad \begin{cases} u_t = u_{xx} + h_1 \\ u(x,0) = f_1 \end{cases} & \quad \implies \quad \sum_n c_n(t)\phi_n \\
\text{IC source} & \quad \implies \quad \begin{cases} u_t = u_{xx} + h_2 \\ u(x,0) = f_2 \end{cases} & \\
\vdots & \\
\end{align*}
\]

Each projected problem is independent of the others, and depends only on the \( n \)-th mode of the inputs (the source and IC).

The full solution is then a superposition of these modes:

\[
u(x,t) = \sum_n u_n(x,t) = \sum_n (\text{solution to IBVP}_n)
\]

**Practical example I:** This principle tells us that if there is no \( \phi_N \) term for some \( N \) in either the source or IC then there will also be no \( \phi_N \) term for the full solution. This is easy to show, since the \( N \)-th projected problem is then

\[
\begin{align*}
(u_N)_t & = -L(u_N) + h_n(t)\phi_n(x) \\
(\text{hom. BCs at } x = a \text{ and } x = b) \\
u_n(x,0) & = f_n\phi_n(x).
\end{align*}
\]

By a guess, the solution to this is just \( u_N(x,t) = 0 \).

Moreover, if there is only one mode in the inputs, then the solution will also have only one mode. For a simple example, consider

\[
\begin{align*}
u_t & = u_{xx} + 4 \sin 2x, \quad x \in (0, \pi), \ t > 0 \\
u(0,t) & = 0, \quad u(\pi,t) = 0, \\
u(x,0) & = 2 \sin 2x.
\end{align*}
\]
The eigenfunctions are $\phi_n = \sin nx$ for $n \geq 1$. Both the source and IC only have a $\phi_2$ term. To be explicit, we have

$$u(x, 0) = 2\phi_2 = \sum_{n \geq 1} f_n \phi_n, \quad f_n = \begin{cases} 2 & n = 2 \\ 0 & n \neq 2 \end{cases}.$$  

It follows that only the $n = 2$ mode of the solution is non-zero (all the other projected problems have $u_n = 0$ as their solution), so

$$u(x, t) = c_2(t)\phi_2(x).$$

Indeed, plugging this into the PDE we find that

$$c_2'(t) = -\lambda_2 c_2(t) + 4, \quad c_2(0) = 2 \implies c_2 = 1 + e^{-4t}.$$  

No other terms need to be solved for ($c_n(t) = 0$ for $n \neq 2$). The solution is simply

$$u(x, t) = (1 + e^{-4t}) \sin 2x.$$

**Practical example II:** Using this and superposition, we can often greatly simplify by splitting the problem into a single-mode part and then the rest. Take, for example,

$$u_t = u_{xx} + 3e^{-t} \sin x, \quad x \in (0, \pi), \ t > 0$$

$$u(0, t) = 0, \ u(\pi, t) = 0,$$

$$u(x, 0) = x(1 - x).$$

The initial condition is not a finite sum of modes, so $u$ will be an infinite series. But the source term is just $t\phi_1$. We can split $u$ into two parts, $u = v + w$ where (writing it out completely just for clarity)

$$v_t = v_{xx} + 3e^{-t} \sin x, \quad w_t = w_{xx},$$

$$v(0, t) = v(\pi, t) = 0, \quad w(0, t) = w(\pi, t) = 0,$$

$$v(x, 0) = 0, \quad w(x, 0) = x(1 - x).$$

The first part only has a $\phi_1$ mode; the solution is

$$v = a_1(t)\phi_1(x), \quad a_1'(t) + a_1(t) = 3e^{-t}, \ c_1(0) = 0 \implies c_1 = 3te^{-t}.$$  

The solution for $w$ is the same as the Dirichlet example from earlier - no source term means

$$w = \sum_{n \geq 1} b_n(t)\phi_n(x), \quad b_n'(t) + \lambda_n b_n(t) = 0 \implies b_n(t) = b_n(0)e^{-\lambda nt}$$

with no case work required to deal with the source. The full solution is then

$$u = 3te^{-t} \sin x + \sum_{n=1}^{\infty} b_n(0)e^{-\lambda nt}\phi_n(x)$$

with $b_n(0)$ determined by the initial condition for $w$. 
3. Separation of variables

For **homogeneous** problems, we can exploit this independence to obtain solutions quickly. Not that what follows is a **useful computational trick**, and is justified because of the theoretical framework of eigenfunctions.

3.1. A first example. Consider the equation

\[ u_t = u_{xx}, \quad x \in [0, \pi], \quad t > 0 \]
\[ u(0, t) = u(\pi, t) = 0, \quad t > 0 \]
\[ u(x, 0) = f(x). \]

We know that the solution will be an infinite sum of terms (modes) of a certain form. Let us guess a **separated** solution

\[ u(x, t) = F(t)G(x). \]

Plug into the PDE to get

\[ F'(t)G(x) = F(t)G''(x). \]

Now separate variables, putting all the \( x \)'s on one side:

\[ \frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)}. \]

The left side is independent of \( x \) and the right side is independent of \( t \). Thus, both must equal the same constant (chosen to be \(-\lambda\), knowing it will be the eigenvalue):

\[ \text{ind. of } x = \text{ind. of } t = \text{const.} \]
\[ \implies \frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = -\lambda. \]

Plugging this into the boundary conditions, we find that

\[ F(t)G(0) = F(t)G(\pi) = 0 \text{ for all } t \]

so we should require

\[ G(0) = G(\pi) = 0. \]

This gives a pair of ODEs, linked by a shared constant, one of which has BCs:

\[ F'(t) = -\lambda F(t), \quad -G''(x) = \lambda G(x), \quad G(0) = G(\pi) = 0. \]

The problem for \( G \) is the eigenvalue problem (for the operator \( Lu = -u_{xx} \)) with solutions

\[ G_n = \sin nx, \quad \lambda_n = n^2 \text{ for } n = 1, 2, \ldots. \]

This sets the possible constants. Now for each constant \( \lambda_n \), solve for \( F \):

\[ \lambda_n \implies F_n = b_n e^{-\lambda_n t}. \]

We have now found all separated solutions:

\[ u = F(t)G(x) \text{ solves the PDE + BCs } \iff u = u_n(x, t) = b_n e^{-\lambda_n t} \sin nx. \]
the full solution is then a superposition of the separated solutions (the theory for the eigen-
function basis is required to verify this claim is true):

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(x, t). \]

Finally, apply the initial conditions to get the constants \( b_n \) (same as before).

**Comparison to eigenfunctions:** This method is equivalent to solving the projected prob-
lem for the \( n \)-th mode by guessing its form, then putting the solutions together.

However, the method requires that the PDE and BCs are homogeneous. For example,

\[ u_t = u_{xx} + h(x, t) \]

cannot be separated; the eigenfunction method is required.

3.2. **Method (summary):** Suppose the PDE and BCs are homogeneous. We find the
'general' solution to the PDE with the BCs as follows:

**Part I (eigenfunctions):**
- Guess a separated solution, a product of functions of each variable:
  \[ u(x, t) = F(t)G(x). \]
- Plug into the PDE and separate independent variables to get
  function of \( t = \) function of \( x \).
  Conclude that both are equal to a shared constant:
  \[ \text{function of } t = \text{function of } x = -\lambda. \]

  **Note:** If this step fails, we must use another method. \(^2\)
- Plug into the (homogeneous) BCs to get conditions on the ODE(s).
- Identify which ODE is the eigenvalue problem: this gives the correct operator \( L \) for
  the problem. Solve it to get eigenvalues/eigenfunctions.

**Part II (complete SoV):**
- Also solve the other ODEs to get general separated solutions.
- Form the full series using superposition,
  \[ u = \sum u_n(x, t). \]
- Apply the remaining ‘non-separable’ conditions that were unused (e.g. ICs) to solve
  for the unknown constants. (This step is the same as for the eigenfunction method).

\(^2\)The choice of a negative sign is made to make \( \lambda \) positive, which requires knowledge of the problem (if
omitted, it just flips the sign of \( \lambda \)).
Important note: Part I can be used to identify the operator $L$ and the eigenvalue problem for use with the eigenfunction method when this is not clear (see next example).

To do so, discard any inhomogeneous terms (sources, etc.) and proceed with Part I of SoV. Then stop and return to the eigenfunction method now that you have the $\phi_n$'s and the correct $L$; you can write e.g.

$$u_t = -Lu + h(x,t), \quad u = \sum_n c_n(t)\phi_n(x)$$

plug in and solve.

A second example: Consider the PDE/BCs

$$u_t = u_{xx} + u_x + (t+1)u + h(x,t),
\quad u(0,t) = 0, \quad u(1,t) + u_x(1,t) = 0$$

Case I (homogeneous): Suppose $h(x,t) = 0$. Then the PDE is homogeneous, so we can proceed with separation of variables to get the solution. Plug in $u = F(t)G(x)$ to get

$$F'(t)G(x) = F(t)G''(x) + F(t)G'(x) + (t+1)F(t)G(x).$$

Divide by $G(x)$ and $F(t)$ to get:

$$\frac{F'(t)}{F(t)} = \frac{G'' + G'}{G} + (t+1).$$

$$\implies \frac{F'(t)}{F(t)} - (t+1) = \frac{G'' + G'}{G} = -\lambda.$$

Now plug into the BCs:

$$F(t)G(0) = 0, \quad F(t)(G(1) + G'(1)) = 0$$

$$\implies G(0) = 0, \quad G(1) + G'(1) = 0.$$

The separated problems are then

$$F' = -(\lambda + t + 1)F$$
$$G'' + G' = -\lambda G, \quad G(0) = G(1) + G'(1) = 0.$$

We then solve this to get eigenfunctions $G_n$ and eigenvalues $\lambda_n$, then solve

$$F' + (\lambda_n + t + 1)F = 0 \implies F_n(t).$$

Now use superposition to write the full solution,

$$u = \sum_n F_n(t)G_n(x).$$

Case II (inhomogeneous): Now suppose that $h(x,t)$ is non-zero. The choice of $L$ may not be clear, so we can use SoV to find it as above, searching for solutions

$$u(x,t) = F(t)\phi(x)$$
(changing notation to match previous examples). Discard \( h(x,t) \) and use SoV on the homogeneous problem to get

\[
\phi'' + \phi' = -\lambda \phi, \quad \phi(0) = 0, \quad \phi(1) + \phi'(1) = 0
\]

which is the correct eigenvalue problem and tells you that

\[
Lu = -u_{xx} - u_x.
\]

Now stop and return to the full problem, now known to be

\[
u_t = -Lu + (t + 1)u + h(x,t).
\]

Proceed with the eigenfunction method; write

\[
u = \sum_n F_n(t) \phi_n(x), \quad h(x,t) = \sum_n h_n(t) \phi_n(x)
\]

and so on and plug in. The method will work (since the operator \( L \) is correct) and yield

\[
F_n'(t) = -\left(\lambda_n + t + 1\right) F_n(t) + h_n(t).
\]

Note that this matches the SoV solution when \( h = 0 \) (as it should, because that is the homogeneous problem), but we could not have gotten the \( h_n(t) \) term with SoV alone.

3.3. Another example. Here is an example (to be solved in detail later) where SoV is useful to make progress in solving an unfamiliar problem. We convert the PDE in two variables to two one dimensional ODEs that are easier to work with.

Consider Laplace’s equation in a disk of radius \( a \) for \( u(r, \theta) \), which is

\[
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{for} \quad \theta \in [0, 2\pi], \ r < a
\]

with a boundary condition on the boundary of the disk,

\[
u(a, \theta) = f(\theta).
\]

Assume a separated solution

\[
u(r, \theta) = R(r)G(\theta)
\]

and substitute into the PDE:

\[
R''G + \frac{1}{r} R'G + \frac{1}{r^2} RG'' = 0.
\]

Divide by \( RG \) to get

\[
\frac{R'' + R'/r}{R} + \frac{1}{r^2} \frac{G''}{G} = 0
\]

and then move terms around to separate the \( r \) and \( \theta \); we end up with

\[
-r^2 \frac{R'' + R'/r}{R} + \frac{G''}{G} = -\lambda.
\]

so the ODEs for \( R \) and \( G \) are

\[
G'' + \lambda G = 0, \quad r^2 R'' + r R' - \lambda R = 0
\]

There are some subtleties to be addressed here - the boundary conditions are not obvious, and our theory does not apply to this problem (the BCs are not standard). This will be addressed later.
4. Wave equation

The wave equation, in one dimension, has the form

\[ u_{tt} = c^2 u_{xx} \]

for \( u(x,t) \). Here \( c \) is the ‘wave speed’. This is the fundamental equation for describing propagation of (physical) waves e.g. electromagnetic, seismic, sonic and so on. As with the heat equation, the wave speed may vary in space. For a vibrating string with variable density \( \rho(x) \) and tension \( T \) (constant), we have, for instance,

\[ \rho(x)u_{tt} = Tu_{xx}. \]

4.1. Vibrating string. Consider, for example a string that is fixed at ends \( x = 0 \) and \( x = L \) with constant tension \( T \) and density \( \rho \). Let \( c = \sqrt{T/\rho} \). Then the displacement \( u(x,t) \) of the string can be described by the wave equation:

\[ u_{tt} = c^2 u_{xx}, \quad x \in (0,L) \]

The derivation is standard (see e.g. the book). Suppose that the string has, at \( t = 0 \), an initial displacement \( f(x) \) and speed \( g(x) \). The IBVP for \( u(x,t) \) is

\[
\begin{align*}
  u_{tt} &= c^2 u_{xx}, \quad x \in (0,L), \ t \in \mathbb{R} \\
  u(0,t) &= 0, \quad u(L,t) = 0, \\
  u(x,0) &= f(x), \quad u_t(x,0) = g(x).
\end{align*}
\]

(4.1)

Note that there are two ICs needed because of the two \( t \)-derivatives. A sketch and the domain (in the \((x,t)\) plane) is shown below. We do not restrict \( t > 0 \) as in the heat equation.

4.2. Solution (separation of variables). Look for a separated solution

\[ u = h(t)\phi(x). \]

Substitute into the PDE and rearrange terms to get

\[ \frac{1}{c^2} \frac{h''(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda. \]

The eigenvalue problem and solution are:

\[ \phi'' + \lambda \phi = 0, \quad \phi(0) = \phi(L) = 0, \]

\[ \implies \phi_n = \sin \frac{n\pi x}{L}, \quad \lambda_n = \frac{n^2\pi^2}{L^2}. \]
Define the fundamental frequency\(^3\) and its multiples
\[ \omega_1 = \pi c / L, \quad \omega_n = n\pi c / L. \]
For each \( \lambda_n \), we solve for the solution \( h_n(t) \) (ICs to be applied later):
\[ h_n'' + c^2 \lambda_n h_n = 0 \quad \implies h_n = a_n \cos \omega_n t + b_n \sin \omega_n t. \]
The full solution to the PDE is then the series
\[ u(x, t) = \sum_{n=1}^{\infty} h_n(t) \phi_n(x) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin \left( \frac{n\pi x}{L} \right). \quad (4.2) \]
To find the coefficients, project onto \( \phi_n \) to get
\[ a_n = h_n(0) = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}, \quad \omega_n b_n = h_n'(0) = \frac{\langle g, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}. \]
Explicitly, the formulas are (note that this is just a Fourier sine series)
\[ a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \quad b_n = \frac{2}{nc\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx. \quad (4.3) \]

4.3. Standing waves. The separated solutions (the ‘modes’) in (4.2) have the form
\[ u_n(t) = (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin(n\pi x / L). \]
These solutions are standing waves, because they have points that stay fixed (‘nodes’). The frequency \( \omega_0 \) is the lowest (natural) frequency of vibration for the string.

Where is the wave? So far, it is not clear why the full solution describes a propagating wave. With some effort we can show that the solution to the wave equation is really a superposition of two superimposed waves traveling in opposite directions. Using
\[ \cos nct \sin nx = \frac{1}{2} (\sin n(x + ct) + \sin n(x - ct)) = \frac{1}{2} h_n(x + ct) + \frac{1}{2} h_n(x - ct) \]
we can rewrite the solution in the form \( F(x + ct) + G(x - ct) \) (D’Alembert’s formula). This hints at key structure for the wave equation (propagation along characteristics) that is outside of the scope of the eigenfunction method; we will not pursue it here.

\(^3\)Definitions vary by a factor of 2\(\pi\); typically \( \omega_0 = c / 2L \) instead.
**Example: plucking a string.** Suppose a string from a guitar or harp is plucked. The initial speed $u_t(x,0) = 0$ and the displacement will a triangular shape like

$$f = A \begin{cases} 2x/L & 0 \leq x < L/2 \\ 2(L-x)/L & L/2 < x < L \end{cases}$$

where $A$ is the initial displacement at $x = L/2$. It is straightforward to show that (with $\omega_n$ as before) the response of the string (Figure 2) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t) \sin \frac{n\pi x}{L}, \quad a_n = \frac{8A}{\pi^2 n^2} \sin \frac{n\pi}{2}.$$

(4.4)

Since $\sin n\pi/2 = 0$ for $n$ even, the string, when plucked exactly at the center, vibrates with only the odd harmonics, and the amplitude decays like $1/n^2$. Note that because the initial displacement is not an eigenfunction, there are an infinite number of harmonics present.

![Figure 2](image_url)

**Figure 2.** Left: solution (4.4) and initial condition (dashed). Right: solution and its two waves $h_n(x \pm ct)$ (red and blue).

### 4.4. Solution via eigenfunctions.

The procedure is similar to SoV; we solve

$$u_{tt} = -c^2 Lu, \quad Lu = -u_{xx}$$

(the $c^2$ can be put inside $L$ but it is easier to keep $L$ simple). Solve the eigenvalue problem

$$-\phi'' = \lambda \phi, \quad \phi(0) = \phi(L)$$

and into the ICs to get

$$f(x) = u(x,0) \implies c_n(0) = f_n, \quad g(x) = u_t(x,0) \implies c_n'(0) = 0.$$
4.5. **With a source term (tuning the string).** Consider a string that is fixed at both ends. The system starts at rest, and is then driven by some external force. The IBVP is

\[
u_{tt} = c^2 u_{xx} + s(x,t), \quad x \in (0, \pi), \ t \in \mathbb{R}
\]

\[
u(0,t) = 0, \ u(\pi, t) = 0,
\]

\[
u(x,0) = 0, \ u_t(x,0) = 0.
\]

(4.5)

Imagine that we are free to control the time-dependence of the force (waving the string) and wish to ‘see’ the eigenfunctions and eigenvalues. We drive the system with an input

\[s(x,t) = A \sin(\omega t) \cdot h(x)\]

and ‘tune’ by adjusting \(\omega\) until it hits eigenvalues. First, we solve the IBVP in general:

**Part I: the eigenfunctions** Drop the source term to get

\[u_{tt} = c^2 u_{xx} \quad \text{(for part I only!)}.
\]

Proceed as in the previous example with \(u = a(t)\phi(x)\); the result is the same:

\[\phi'' = \lambda \phi, \quad \phi(0) = \phi(\pi) = 0
\]

\[\Rightarrow \phi_n = \sin nx, \quad \lambda_n = n^2, \quad n \geq 1.
\]

Because the full problem is inhomogeneous, **stop** and return to the eigenfunction method.

**Part II: solving the PDE (eigenfunction method):** First, write the source in the eigenfunction basis by factoring out the \(t\) part and expanding \(h(x)\):

\[s(x,t) = A \sin \omega t \sum_{n=1}^{\infty} h_n \phi_n(x), \quad h_n = \frac{\langle h, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{\pi} \int_0^\pi h(x) \phi_n(x) \, dx \quad \text{(4.6)}
\]

Now write the solution in terms of the eigenfunction basis,

\[u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x),
\]

and plug into the PDE to get

\[\sum_{n=1}^{\infty} a_n''(t) \phi_n(x) = c^2 \sum_{n=1}^{\infty} a_n(t) \phi_n''(x) + \sum_{n=1}^{\infty} Ah_n \sin \omega t \phi_n(x)
\]

\[\Rightarrow \sum_{n=1}^{\infty} a_n''(t) \phi_n(x) = \sum_{n=1}^{\infty} (-c^2 \lambda_n a_n(t) + Ah_n \sin \omega t) \phi_n(x)
\]

Now for the initial conditions, plug the series for \(u\) in to find that

\[
\begin{align*}
0 = u(x,0) & \quad \Rightarrow \quad a_n(0) = 0 \text{ for all } n \\
0 = u_t(x,0) & \quad \Rightarrow \quad a_n'(0) = 0 \text{ for all } n
\end{align*}
\]

Putting this together, the coefficient of the \(n\)-th mode \(a_n(t)\) solves

\[a_n''(t) + \omega_n^2 a_n(t) = s_n(t), \quad a_n(0) = a_n'(0) = 0
\]

(4.7)

where \(\omega_n = cn\) is the \(n\)-th ‘fundamental frequency’. The solution is then

\[u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x), \quad a_n(t) = \text{ the solution to (4.7)}.
\]
You can solve the ODE in general (undetermined coefficients), but some illustrative cases will suffice to understand how the solution is affected by the source:

**Case 1 (single mode):** Suppose

\[ s(x,t) = A \sin \omega t \sin 2x. \]

Then \( s \) is a multiple of the \( n = 2 \) eigenfunction \( \phi_2 = \sin(2x) \). The ICs are zero, so by the independence of modes, the other modes are zero, so

\[ u(x,t) = a_2(t) \phi_2(x), \quad a_2'' + c^2 \omega_2^2 a_2 = A \sin \omega t, \quad a_2(0) = a_2'(0) = 0 \]

There are two cases to consider (if \( \sin \omega t \) is a solution to the homogeneous ODE or not):

\[
\begin{align*}
\omega \neq \omega_2 & \implies a_2(t) = \frac{A}{\omega_2 - \omega} \left( \sin \omega t - \frac{\omega}{\omega_2} \sin \omega_2 t \right) \\
\omega = \omega_2 & \implies a_2(t) = \frac{1}{2 \omega_2} \left( \sin \omega_2 t - t \cos \omega_2 t \right)
\end{align*}
\]

There is resonance (unbounded growth) if \( \omega = \omega_2 \), i.e. if the forcing frequency matches the natural frequency of the forcing mode. Otherwise, the solution stays bounded (no growth).

**Case 2 (infinite modes):** Now suppose \( s(x,t) = \sin \omega t \) (uniform in \( x \)). Then

\[ s(x,t) = \sin \omega t = \sin \omega t \sum_{n=1}^{\infty} f_n \phi_n(x), \quad f_n = \frac{2}{\pi} \int_0^\pi \phi_n(x) \, dx = \frac{2}{n \pi} (1 - \cos n \pi). \]

The values \( f_n \) are non-zero for odd \( n \), so \( s \) contains all odd modes. Thus,

\[ a_n''(t) + \omega_n^2 a_n(t) = f_n \sin \omega t, \quad \text{for } n \text{ odd} \]

\[ a_n(t) = 0 \quad \text{for } n \text{ even} \]

For each \( n \), the solution works as in (4.8) of Case 1. It follows that

- If \( \omega = \omega_N \) for some \( N \), the forcing drives the \( N \)-th mode to resonance, while the other modes stay bounded, and

\[ u(x,t) \sim a_n(t) \phi_N(x) \text{ as } t \text{ grows, with } \max_t |a_n(t)| \sim Ct. \]

The max. amplitude of this resonant term grows like \( t \), and so this term is eventually arbitrarily large relative to the other terms; then after a long time, we can observe the eigenfunction \( \phi_N(x) \). Examples for \( \omega = 3 \) and 5 are shown below.

- If \( \omega \) is not equal to any of the natural frequencies \( \omega_n \), no mode is resonant; all stay bounded so \( u(x,t) \) stays bounded. No single term grows arbitrarily large.
5. INTERPRETING SOLUTIONS

The convergence of Fourier series provides some insight into solutions. Consider

$$u_t = u_{xx}, \quad x \in (0, \pi), \ t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0,\quad u(x, 0) = x/\pi$$

(5.1)

for the heat equation. Over time, the solution will spread out, with heat leaking out of the boundaries, and $u \to 0$. Curiously, we can get a solution to (5.1) even though the BCs are not satisfied by the initial condition at $x = \pi$:

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \sin nx, \quad n \geq 1 \quad (5.2)$$

which satisfies the PDE and the BCs, and from the initial condition,

$$x = \sum_{n=1}^{\infty} a_n \sin nx \implies a_n = \frac{2}{\pi^2} \int_{0}^{\pi} x \cdot \sin nx \, dx = \frac{2}{n\pi} (-1)^{n+1} \neq 0 \text{ for all } n.$$

Note that (5.2) at $t = 0$ is the sine series for $x$ in $[0, \pi]$. As the solution evolves in $t$, there are three notable types of behavior for the solution:

- **At $t = 0$ exactly:** The series converges to $x$ for $x \in (0, \pi)$ including zero but not at $x = \pi$. Thus, at $t = 0$, there is a jump at $x = \pi$ that causes Gibbs’ phenomenon when using an approximation with a finite number of terms (see below).
- **For $t > 0$, small $t$:** However, for $t > 0$, the solution is smooth. The jump at $x = \pi$ is quickly ‘smoothed out’ - the discontinuity becomes a sharp slope, then evens out.
- **For large $t$:** After the initial smoothing phase, the first term is dominant (much larger than the others so $u(x, t)$ is $\approx$ a multiple of $\phi_1$ and decays exponentially.

Our ‘eigenfunction expansion’ (5.2) is a solution in the Fourier sense, i.e. the series converges according to that theory. But $u(x, t)$ is smooth except at $t = 0, x = \pi$. 

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![Graphs showing solution evolution](image-url)
What about the wave equation? In contrast, the series for the wave equation does not have this nice property: the waves propagate, but do not spread out. To compare, consider the same IBVPs for the wave and heat equation with a jump placed at the midpoint:

\[ u_t = u_{xx}, \quad x \in (0, \pi), \]
\[ u(0, t) = u(\pi, t) = 0, \] \hspace{1cm} (H)
\[ u(x, 0) = \begin{cases} 
    x & x < \pi/2 \\
    0 & x > \pi/2
\end{cases} \]

\[ u_{tt} = u_{xx}, \quad x \in (0, \pi), \]
\[ u(0, t) = u(\pi, t) = 0, \] \hspace{1cm} (W)
\[ u(x, 0) = \begin{cases} 
    x & x < \pi/2 \\
    0 & x > \pi/2
\end{cases}, \]
\[ u_t(x, 0) = 0 \]

- As discussed, the jump at \( x = \pi/2 \) (the only jump) gets smoothed out for the heat equation; after \( t > 0 \) the solution / series approximation for (H) are smooth.
- The jump in the wave equation travels across the domain (a wave!). It does not get smoothed out - the discontinuity in \( u(x, t) \) persists forever. The eigenfunction series for (W) always converges to the average at the jump and has Gibbs' phenomenon.

**Important principle (smoothing):** This example illustrates that the heat equation smooths out initial data. When there is a sharp gradient, diffusion will act quickly to reduce the jump (everyday example: open an oven; the initial blast of hot air is evening out the temperature between the oven and the kitchen).

The wave equation, in contrast, does not smooth initial data. The data is ‘transported’ (wave propagation) through the domain, but does not spread out. Thus, the fact that the eigenfunction series converges ‘in the Fourier’ sense matters at all \( t \).