MATH 356 LECTURE NOTES
INTRO TO PDES
IBVPS AND EIGENVALUE PROBLEMS

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Topics covered

- The heat equation
  - Definitions: initial boundary value problems, linearity
  - Types of boundary conditions, linearity and superposition
- Eigenfunctions
  - Eigenfunctions and eigenvalue problems; computation
  - Standard examples: Dirichlet and Neumann
  - Structure of the eigenfunctions/eigenvalues
- More properties of eigenfunctions
  - Oscillation (why is this required)
  - Why is the set of eigenvalues discrete?
  - Proof of orthogonality
  - Rayleigh quotient (show $\lambda \geq 0$)

1. The heat equation: preliminaries

Let $[a, b]$ be a bounded interval. Here we consider the PDE

$$u_t = u_{xx}, \quad x \in (a, b),\ t > 0. \tag{1.1}$$

for $u(x, t)$. This is the **heat equation** in the interval $[a, b]$.

Remark (adding a coefficient): More generally, we could consider

$$u_t = k u_{xx}$$

where $k > 0$ is a 'diffusion coefficient'. However, since the constant can be scaled out by defining a rescaled time $\tau = t/k$ to get

$$u_\tau = u_{xx}$$

there is no loss of generality in studying the structure of (1.1). Note that it is **essential that the coefficient is positive**; a negative k will produce drastically different results.

1.1. Initial and boundary conditions.
An **initial boundary value problem** (IBVP) for the heat equation consists of the PDE itself plus three other conditions specified at

$x = a, x = b$ and $t = 0$. As a simple example:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad t > 0 \text{ and } x \in (a, b), \tag{1.2a}$$

$$u(a, t) = 0 \text{ and } u(b, t) = 0 \qquad \text{for } t > 0 \tag{1.2b}$$

$$u(x, 0) = f(x). \tag{1.2c}$$

There are three components:

The PDE: Equation (1.2a) is the PDE (sometimes just 'the equation'), which thThe be solution must satisfy in the entire domain ($x \in (a, b)$ and $t > 0$ here).

Boundary conditions (BCs): Equations (1.2b) are the **boundary conditions**, imposed at the x-boundaries of the interval. Each BC is some condition on u at the boundary.

Initial conditions (ICs): Equation (1.2c) is the **initial condition**, which specifies the initial values of u (at the initial time $t = 0$).

The initial boundary value problem (1.2a)-(1.2c) has a unique solution provided some technical conditions hold on the boundary conditions.

One can think of the 'boundary' of the solution domain to have three sides: $\{x = a\}, \{x = b\}$ and $\{t = 0\}$, with the last side left open (the solution fills this in as $t \to \infty$). The initial condition is really a boundary condition at $t = 0$.[1]
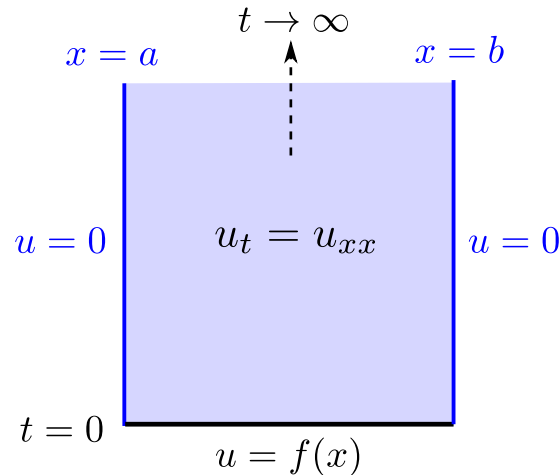
Definition (important BCs): There are three basic types of boundary conditions. Most of the time, we will consider one of these when solving PDEs.

Dirichlet: $u(a, t) = 0$ (or 'zero boundary conditions')

Neumann: $u_x(a, t) = 0$ (or 'zero flux')

Robin: $\alpha u_x(a, t) + \beta u(a, t) = 0$ (or 'radiation')

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[1]The three-sided boundary is called the **parabolic boundary** of the IBVP.

The heat equation could have different types of boundary conditions at a and b, e.g.

$$u_t = \alpha u_{xx}, \quad x \in [0,1], \ t > 0$$
$$u(0,t) = 0, \quad u_x(1,t) = 0$$

has a Dirichlet BC at $x = 0$ and Neumann BC at $x = 1$.

Modeling context: For the heat equation $u_t = \alpha u_{xx}$, these have physical meaning. Recall that u is the temperature and $-\alpha u_x$ is the heat flux.

Dirichlet: The temperature u is fixed at the end.

Neumann: The end is insulated (no heat enters or escapes).

Radiation: Some heat enters or escapes, with an amount proportional to the temperature:

$$-\alpha u_x = \beta u.$$

For the interval $[a, b]$ whether heat enters or escapes the system depends on the endpoint and β. The heat flux $-\alpha u_x$ is to the right if it is positive, so at the left boundary a, heat enters the system when $\beta > 0$ and leaves when $\beta < 0$.

Similarly, at the right boundary b, heat enters the system when $\beta < 0$ and leaves when $\beta > 0$.

The same interpretations apply when the equation is describing diffusion of some other quantity (e.g. diffusion of a chemical in a tube).

1.2. **Linearity and homogeneous PDEs.** The definitions of linear and homogeneous extend to PDEs. We call a PDE for $u(x,t)$ **linear** if it can be written in the form

$$L[u] = f(x,t)$$

where f is some function and L is a linear operator involving the partial derivatives of u. Recall that **linear** means that

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2].$$

The PDE is **homogeneous** if $f = 0$ (so $l[u] = 0$) and **inhomogeneous** if f is non-zero.

Some examples of linear PDEs we will study are

$$u_t = u_{xx} + g(x,t) \qquad (L[u] = u_t - u_{xx}),$$
$$u_{tt} = u_{xx} + g(x,t) \qquad (L[u] = u_{tt} - u_{xx}),$$
$$u_{xx} + u_{yy} = g(x,y) \qquad (L[u] = u_{xx} + u_{yy} = \nabla^2 u),$$

which are the heat equation, wave equation and the Poisson equation, respectively. Note that the function is $u(x,y)$ in the last one. An example of a non-linear PDE would be

$$u_t + u u_x = u_{xx}$$

The same definitions apply to **boundary conditions**. All the boundary conditions listed in the previous section are linear homogeneous. For example,

$$u_x(a,t) = 0 \qquad\qquad (1.3)$$

is a linear boundary condition since if u and v satisfy (1.3) and

$$w = c_1 u + c_2 v$$

then w also satisfies (1.3) since

$$w_x(a, t) = c_1 u_x(a, t) + c_2 v_x(a, t) = 0.$$

Non-homogeneous boundary conditions can be imposed, for instance

$$u(a, t) = t$$

which might be used to model the ambient temperature increasing with time.

Key fact: A linear, homogeneous PDE obeys the **superposition principle**:

$$u_1, u_2 \text{ are solutions} \implies c_1 u_1 + c_2 u_2 \text{ is a solution} \qquad (1.4)$$

for all scalars $c_1, c_2 \in \mathbb{R}$. The same definition applies to boundary conditions. For instance, all the boundary conditions listed above are linear homogeneous.

Note that an inhomogeneous PDE does not have this property! However, the 'homogeneous' part (i.e. the equation with the inhomogeneous term set to zero) does, and we will find that, as with ODEs, superposition will still be useful.

1.3. **More on superposition.** The superposition principle (1.4) is a crucial feature of linear homogeneous problems. Note that while this property is true for homogeneous PDEs and boundary conditions, it is not quite true when initial conditions are included. If u and v are both solutions to the homogeneous problem

$$u_t = u_{xx}, \qquad t > 0 \text{ and } x \in (a, b),$$
$$u(a, t) = u(b, t) = 0 \quad \text{for } t > 0$$

where u has initial condition

$$u(x, 0) = f_1(x)$$

and v has initial condition

$$v(x, 0) = f_2(x)$$

then $w = u + v$ solves the IBVP

$$w_t = w_{xx}, \qquad t > 0 \text{ and } x \in (a, b),$$
$$w(a, t) = w(b, t) = 0 \quad \text{for } t > 0$$
$$w(x, 0) = f_1(x) + f_2(x).$$

Superimposing two solutions to the PDE with BCs will give another solution, and the initial conditions get superimposed.

We can exploit superposition to split a problem into simpler parts. For example, suppose we seek u solving the inhomogeneous problem

$$u_t = u_{xx} + h(x, t),$$
$$u(0, t) = u(1, t) = 0,$$
$$u(x, 0) = f(x)$$

with non-zero initial conditions. We can split this into

$$u = v + w$$

where v solves an *inhomogeneous* problem with *zero* initial conditions and $w(x,t)$ solves a *homogeneous* problem with *non-zero* initial conditions,

$$v_t = v_{xx} + h(x,t), \qquad\qquad w_t = w_{xx},$$
$$v(0,t) = v(1,t) = 0, \qquad\qquad w(0,t) = w(1,t) = 0,$$
$$v(x,0) = 0 \qquad\qquad w(x,0) = f(x),$$

thereby splitting the problem for u into two simpler parts.

To check this, plug $v + w$ into the PDE, boundary conditions and initial conditions and use linearity. For the PDE, we check that

$$u_t = (v + w)_t = v_t + w_t = v_{xx} + w_{xx} + h(x,t) = u_{xx} + h(x,t).$$

For the boundary at $x = 0$, we have

$$u(0,t) = v(0,t) + w(0,t) = 0$$

and similarly $u(1,t) = 0$. Finally, for the initial condition,

$$u(x,0) = v(x,0) + w(x,0) = 0 + f(x) = f(x).$$

2. EIGENFUNCTIONS

Return to the heat equation in a bounded domain with Dirichlet boundary conditions:

$$u_t = u_{xx}, \quad x \in (a,b), \quad t > 0 \tag{2.1}$$
$$u(a,t) = u(b,t) = 0, \quad t > 0. \tag{2.2}$$

Now write the PDE in the form

$$u_t = -L[u], \tag{2.3}$$

where L is the linear operator

$$L[u] = -u_{xx}$$

or, in operator notation,

$$L = -\frac{\partial^2}{\partial x^2}.$$

Now let us regard L as an operator acting on functions $\phi(x)$ (just functions of x), i.e.

$$L[\phi] = \phi'' \text{ or } L = -\frac{d^2}{dx^2}.$$

Definition (eigenfunction): We say that ϕ is an **eigenfunction** of the problem (2.1) with boundary conditions (2.2) if it solves the **eigenvalue problem**

$$L[\phi] = \lambda\phi, \quad \phi(a) = 0, \phi(b) = 0 \tag{2.4}$$

for some $\lambda \in \mathbb{R}$ (the 'eigenvalue'). Equivalently, we say that ϕ is an eigenfunction of the operator L on $[a,b]$ with boundary conditions $\phi(a) = 0$, $\phi(b) = 0$.

Note: the boundary conditions (2.2) can be replaced with some other conditions; the

definition is the same. The boundary conditions for ϕ are the result of plugging $u = \phi(x)$ into the boundary conditions for u.

Important note: To have an eigenfunction of the operator L, we must prescribe an interval $[a, b]$ and associated boundary conditions.

For instance, for the problem

$$u_t = u_{xx}, \quad x \in (0, \pi) \quad t > 0,$$
$$u(0, t) = u_x(\pi, t) = 0, \quad t > 0,$$

the eigenvalue problem is

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi'(\pi) = 0$$

and we say ϕ is an eigenfunction for $L = -d^2/dx^2$ on $[0, \pi]$ with Neumann boundary conditions (or, explicitly, 'with boundary conditions $\phi'(0) = \phi'(\pi) = 0$').

Notice that the eigenvalue problem is an **ODE**, so we are really studying a type of ODE problem and making use of it (later) to solve PDEs. In the following section, we forget about the PDE part for now and find some eigenfunctions in typical cases.

Notation (why the negative sign?): The negative sign is just convention and is not necessary; we could instead write

$$u_t = L[u], \qquad L = \frac{\partial^2}{\partial x^2}.$$

The ODE for the eigenvalue problem is then

$$\phi'' = \lambda\phi$$

which is the same as the eigenvalue problem for $L = -\partial^2/\partial x^2$, with the sign of λ reversed since

$$-\phi'' = (-\lambda)\phi.$$

Since λ **is an unknown anyway**, it does not matter; we will just get λ's that differ by a negative sign and the eigenfunctions will be the same.

The reason for using the negative sign is that it tends to make most, if not all, the eigenvalues positive (rather than mostly/all negative); see examples below.

3. Solving the eigenvalue problem

An operator L in $[a, b]$ with **homogeneous** boundary conditions has an associated **eigenvalue problem** to find an eigenfunction ϕ in $[a, b]$ and an eigenvalue λ such that

$$L\phi = \lambda\phi, \qquad \text{(hom. BCs for } \phi) \tag{3.1}$$

Procedure for eigenvalue problems: The general procedure for solving the eigenvalue problem (3.1) is

a) In each range of λ where the DE has a certain form, find the general solution

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n$$

where n is the order of the DE, using standard ODE solving techniques.

b) Use the boundary conditions $B\phi = 0$ to get $\approx n$ equations for the c's (plus any other constraints relevant to the problem from elsewhere)

c) Find all λ such that there are non-trivial solutions (c's not all zero) and identify the eigenfunction (basis for each set of solutions to $L\phi = \lambda\phi$)

- For (c), often the equation has the form $A(\lambda)\mathbf{c} = 0$ so the condition for an eigenvalue λ is that $\det A(\lambda) = 0$ (to get a non-trivial solution)

There are three standard examples. Consider the operator

$$Lu = -\frac{d^2u}{dx^2}$$

in $[0, \pi]$ with three different boundary conditions. The eigenfunctions should look familiar.

3.1. **Dirichlet BCs.** The eigenvalue problem for $\phi(x)$ is

$$-\phi'' = \lambda\phi, \ x \in (0, \pi), \qquad \phi(0) = 0, \quad \phi(\pi) = 0.$$

To solve it, find the general solution to the DE, then use the boundary conditions and look for values of λ such that a non-trivial solution exists.

Case 1: $\lambda < 0$. We will show here that no solutions exist. The characteristic polynomial is

$$r^2 + \lambda = 0.$$

which has roots $\pm\mu$ where $\mu = \sqrt{-\lambda}$. Then the general solution is

$$\phi = c_1 e^{x\mu} + c_2 e^{-x\mu}.$$

Apply the boundary conditions to get a system for coefficients c_1, c_2:

$$0 = c_1 + c_2, \quad 0 = c_1 e^{\pi\mu} + c_2 e^{-\pi\mu}.$$

This system, in matrix form, is

$$\begin{pmatrix} 1 & 1 \\ e^{\pi\mu} & e^{-\pi\mu} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It has a non-trivial solution if and only if the determinant is zero. But

$$\det(\cdots) = e^{-\pi\mu} - e^{\pi\mu} = 2\sinh(\pi\mu) > 0,$$

so there is only the trivial solution (i.e. $c_1 = c_2 = 0$ is the only solution) for all μ. We conclude there are no negative eigenvalues.

Case 2: $\lambda = 0$. The general solution is $\phi = c_1 x + c_2$. Applying the boundary conditions, we need $b = c_2$ and $c_1 \pi + c_2 = 0$ which forces $c_1 = c_2 = 0$. So again, no eigenvalues in this case.

Case 3: $\lambda > 0$. The general solution is

$$\phi = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x).$$

From the boundary conditions,

$$c_2 \cos 0 = 0$$

$$c_1 \sin(\sqrt{\lambda} \pi) + c_2 \cos(\sqrt{\lambda} \pi) = 0$$

so $c_2 = 0$. To have a non-trivial solution ($c_1 \neq 0$) we need

$$\sin(\sqrt{\lambda} \pi) = 0.$$

This has a non-trivial solution when

$$\sqrt{\lambda} \pi = n\pi, \qquad n = 1, 2, \cdots$$

i.e. for eigenvalues $\lambda_n = n^2$ (note that λ had to be positive to get here, so the $-$ root is discarded!). Plugging back into the general solution (recall that $c_2 = 0$) we obtain the corresponding eigenfunctions

$$\phi_n = \sin nx.$$

Summary: Collecting the results of the three cases, we gfind that only the $\lambda > 0$ case yields eigenvalues. The total set of eigenvalues/functions is

$$\lambda_n = n^2, \ \phi_n = \sin nx, \qquad n = 1, 2, 3, \cdots,$$

which is exactly the basis from the Fourier sine series.

3.2. **Neumann BCs.** The eigenvalue problem is

$$-\phi'' = \lambda \phi, \ x \in (0, \pi), \qquad \phi'(0) = 0, \quad \phi'(\pi) = 0.$$

Case 1: $\lambda < 0$. Again let $\mu = \sqrt{-\lambda}$. The general solution is the same as before, and the boundary conditions require (check this!)

$$0 = \sqrt{\mu}(c_1 - c_2), \qquad 0 = \sqrt{\mu}\left(c_1 e^{\pi \mu} - c_2 e^{-\pi \mu}\right).$$

Again write

$$\begin{pmatrix} 1 & -1 \\ e^{\pi \mu} & -e^{-\pi \mu} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant is $-2\mu \sinh(\pi \mu) < 0$. Note that $e^{\pi \mu} > 1$ and $e^{-\pi \mu} < 1$ since $\mu > 0$. Thus there are still no solutions.

Case 2: $\lambda = 0$. The general solution is

$$\phi = c_1 x + c_2.$$

The boundary conditions require only that $c_1 = 0$, so $\phi = 1$ is an eigenfunction for $\lambda = 0$.

Case 3: $\lambda > 0$. The general solution is the same as before; the boundary conditions require

$$c_1 = 0, \qquad -\sqrt{\lambda}c_2 \sin(\sqrt{\lambda}\pi) = 0.$$

Thus we need $\sqrt{\lambda}\pi = n\pi$, so the eigenvalues are

$$\lambda_n = n^2, \quad n = 1, 2, \cdots$$

with corresponding eigenfunctions

$$\phi_n = \cos nx$$

So in summary, the eigenfunctions/values are

$$\lambda_n = n^2, \qquad \phi_n = \cos nx, \qquad n = 0, 1, 2, \cdots.$$

which are the basis functions for the Fourier cosine series. Note that $n = 0$ is included here (from Case 2), which is absent from the Dirichlet version.

3.3. **Periodic BCs.** Take the domain to be $[0, 2\pi]$ for reasons to be clear. The problem is

$$-\phi'' = \lambda\phi \ x \in (0, 2\pi), \quad \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi).$$

For the details, see HW. The result is that, for $\lambda = \mu^2$,

$$\cos(2\pi\mu) = 1$$

which occurs for the values $\mu_n = n$ for $n = 0, 1, 2, \cdots$. We find that the eigenfunctions are

$$\phi = c_1 \cos nx + c_2 \sin nx$$

which are solutions for all $n \geq 0$ with eigenvalue $\lambda_n = n^2$. Unlike the Dirichlet and Neumann problems, **both terms are allowed**. This means we have eigenfunctions

$$\phi_n = \cos nx, \quad \psi_n = \sin nx \text{ with } \lambda_n = n^2 \text{ for } n = 1, 2, \cdots$$

This is, of course, the basis for the full Fourier series (computed on $[0, 2\pi]$ rather than $[-\pi, \pi]$, but it is the same up to this translation).

Notes on the mechanics: Some points to note when computing eigenfunctions:
- If ϕ is an eigenfunction so is any scalar multiple since eigenvalue problems are always linear homogeneous. We always end up multiplying by an arbitrary constant later, so it does not matter which multiple you choose (e.g. $\sin(n\pi x)$ or $2\sin(n\pi x)$.
- Setting $\lambda = \mu^2$ is just to avoid writing $\sqrt{-\lambda}$ or $\sqrt{\lambda}$.
- The cases are usually but **not always** λ positive/zero/negative.

4. Eigenfunctions and orthogonal bases

The eigenfunctions for $L = -d^2/dx^2$ with any of the standard homogeneous BCs are similar. The BCs all have the form

$$\alpha\phi(a) + \beta\phi'(a) = 0 \tag{4.1}$$

e,g. Dirichlet means $\beta = 0$ and Neumann means $\alpha = 0$.

It turns out that the eigenfunctions are - under suitable technical conditions - an orthogonal basis for functions on $[a, b]$. One version of the theorem[2] is as follows (we'll generalize later):

Theorem on eigenfunctions: Consider the operator

$$L = -\frac{d^2}{dx^2}$$

in an interval $[a, b]$ and the eigenvalue problem

$$L\phi = \lambda\phi, \quad \text{with BCs of the form (4.1) at } x = a \text{ and } x = b. \tag{4.2}$$

Then the set of eigenfunctions $\{\phi_n\}$ solving (4.2) forms an orthogonal basis for $L^2[a, b]$ Moreover, it is true that

   i) There is one eigenfunction for each eigenvalue,

  ii) There are infinitely many positive eigenvalues that increase to ∞,

 iii) There are finitely many negative eigenvalues.

We call the set of eigenvalues the **spectrum** of the operator L (in the interval $[a, b]$ with the associated BCs).

This theorem is a special case of the more general **Sturm-Liouville theory**, which extends to other operators. The key result is that the eigenfunctions form an orthogonal basis, which permits us to represent functions in this nice basis.

Key point (eigenfunction expansion): The theorem says that any (reasonable) function $f(x)$ has a unique representation as an 'eigenfunction expansion',

$$f = \sum_n f_n\phi_n, \quad f_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2\,dx}. \tag{4.3}$$

where $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$ is the L^2 inner product.

The proof is the same as for orthogonal bases of vectors in $\mathbb{R}^n$. We have

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \text{ for } m \neq n.$$

Now take the $\langle \cdot, \phi_n \rangle$ of the expression for f to get (using a dummy sum variable m)

$$\langle f, \phi_n \rangle = \langle \sum_m f_m\phi_m, \phi_n \rangle = \sum_m f_m\langle \phi_m, \phi_n \rangle = f_n\langle \phi_n, \phi_n \rangle.$$

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[2]Note: the theorem is a consequence of a **spectral theorem** for L^2-like spaces, a rather deep result in analysis/linear algebra that generalizes of the spectral theorem for symmetric matrices. As it is well beyond the scope of the course, you will have to take this result on faith.

5. Properties of eigenfunctions

First, an analogy from linear algebra. Let $A \in \mathbb{R}^{n \times n}$ be a real **symmetric** matrix in $\mathbb{R}^n$, i.e. $A^T = A$ and recall that the 'inner product' is the dot product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}.$$

An important property is that **eigenvectors for distinct eigenvalues are orthogonal**.

Proof. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors for eigenvalues λ_1 and λ_2. Then

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$
$$A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$$

Now take the inner (dot) product of the first line with $\mathbf{v}_2$ and of the second line with $\mathbf{v}_1$:

$$(A\mathbf{v}_1)^T \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^T \mathbf{v}_2$$
$$\mathbf{v}_1^T (A\mathbf{v}_2) = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \tag{5.1}$$

But A is symmetric so the left hand sides are equal:

$$A\mathbf{v}_1^T \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 = \mathbf{v}_1 A \mathbf{v}_2. \tag{5.2}$$

It follows from (5.1) that

$$\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \implies (\lambda_1 - \lambda_2)\mathbf{v}_1^T \mathbf{v}_2 = 0.$$

If the eigenvalues are distinct, $\lambda_1 \neq \lambda_2$ so $\mathbf{v}_1^T \mathbf{v}_2 = 0$ (i.e. the eigenvectors are orthogonal). $\square$

Extending the idea to eigenfunctions: The eigenvalue problem for $Lu = -u_{xx}$ has the same property. Consider an interval $[a, b]$ and, for simplicity, the eigenvalue problem

$$-\phi'' = \lambda \phi \text{ for } x \in (a, b), \quad \phi(a) = \phi(b) = 0.$$

Suppose ϕ_1 and ϕ_2 are eigenfunctions with eigenvalues λ_1 and λ_2:

$$-\phi_1'' = \lambda_1 \phi_1$$
$$-\phi_2'' = \lambda_2 \phi_2 \tag{5.3}$$

Take the inner product $\int_a^b \cdot \phi_2 \, dx$ with the first equation and $\int_a^b \cdot \phi_1 \, dx$ with the second to get

$$-\int_a^b \phi_1'' \phi_2 \, dx = \lambda_1 \int_a^b \phi_1 \phi_2 \, dx$$
$$-\int_a^b \phi_1 \phi_2'' \, dx = \lambda_2 \int_a^b \phi_1 \phi_2 \, dx \tag{5.4}$$

We can do a calculation analogous to (5.2) to show the left sides are equal. The trick here is to **integrate by parts twice** and then use the boundary conditions to simplify:

$$-\int_a^b \phi_1'' \phi_2 \, dx = -\phi_1' \phi_2 \Big|_a^b + \int_a^b \phi_1' \phi_2' \, dx$$

$$= (\phi_1 \phi_2' - \phi_1' \phi_2) \Big|_a^b - \int_a^b \phi_1 \phi_2'' \, dx$$

which is **Green's identity**. But ϕ_1 and ϕ_2 vanish at $x = a, b$ so the boundary term is zero:

$$-\int_a^b \phi_1'' \phi_2 \, dx = -\int_a^b \phi_1 \phi_2'' \, dx. \tag{5.5}$$

In the inner product notation (and the operator $L\phi = -\phi''$), this reads

$$\langle L\phi_1, \phi_2 \rangle = \langle \phi_1, L\phi_2 \rangle$$

which is a property of L analogous to what we saw for a symmetric matrix. We'll address this property again later (it turns out to be the property that gives L its essential structure).

Completing the proof of orthogonality, by the inner-product-ed equations (5.4) and (5.5),

$$(\lambda_1 - \lambda_2) \int_a^b \phi_1 \phi_2 \, dx = 0 \implies \int_a^b \phi_1 \phi_2 \, dx = 0$$

so the eigenfunctions are orthogonal if the eigenvalues are distinct ($\langle \phi_1, \phi_2 \rangle = 0$).

For other BCs: As long as the boundary conditions cause the boundary term $(\phi_1 \phi_2' - \phi_1' \phi_2$ to vanish, this proof carries through. This will be true for and of the standard **homogeneous** BCs (left as an exercise).

Summary (orthogonality) For any of the standard **homogeneous** BCs and the eigenvalue problem

$$-\phi'' = \lambda \phi, \; x \in (a, b), \quad \text{(BCs)},$$

the eigenfunctions for distinct eigenvalues are orthogonal:

$$\lambda_i \neq \lambda_j \implies 0 = \langle \phi_i, \phi_j \rangle = \int_a^b \phi_i \phi_j \, dx$$

which we can show by taking the inner product of each eigenvalue problem with the other eigenfunction and **integrating by parts twice**. The crucial property is **Green's identity**

$$-\int_a^b u'' v \, dx = (uv' - u'v) \Big|_a^b - \int_a^b u v'' \, dx$$

which is, abstractly, the condition that the operator L is **symmetric**:

$$\langle Lu, v \rangle = \langle u, Lv \rangle \text{ for all } u, v \text{ satisfying the BCs}.$$

(This observation will be used in full when we consider other operators L).
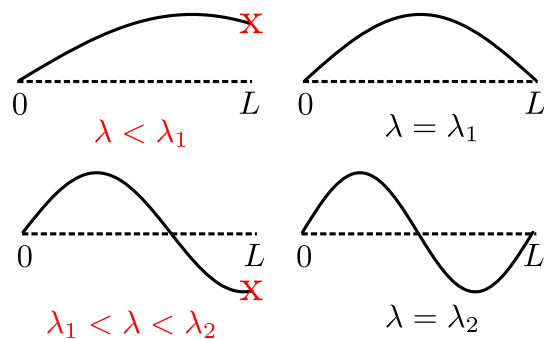
5.1. **Oscillation.** The standing waves example suggests some intuition for understanding the eigenvalue problem. Consider, for simplicity,

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(\pi) = 0.$$

For a solution to the ODE,

$$\phi = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x,$$

to satisfy the BCs, it has to 'fit' in the domain while being zero at the endpoints. The functions sin and cos oscillate, hitting zero at the endpoints only for certain frequences $\sqrt{\lambda_n}$.

The frequency must be such that the solution (the eigenfunction) ϕ oscillates an integer number of times from zero back to zero. This is why the eigenvalues $\lambda_n = n^2$ form a **discrete** set.

The same is true of other **homogeneous** BCs, but the condition it has to meet after each oscillation is different. This is shown below for Neumann BCs ($\lambda_n = (n\pi/L)^2$ and $\phi_n = \cos(n\pi x/L)$ and for Dirichlet at one end, Neumann at the other ($\lambda_n = ((n-1/2)\pi/L)^2$ and $\phi_n = \sin((n-1/2)\pi x/L)$):
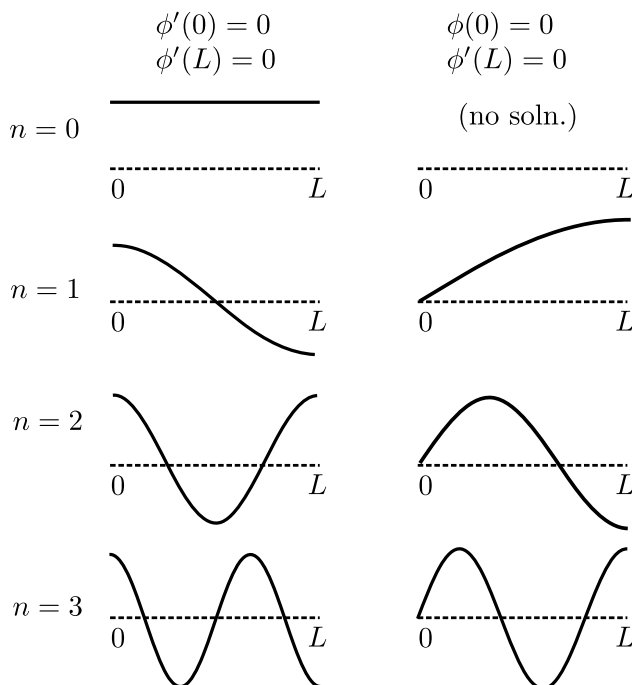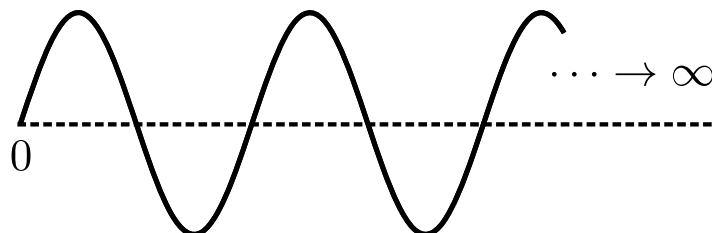
For contrast, consider attempting to solve the eigenvalue problem in an infinite domain:
$$-\phi'' = \lambda\phi \text{ for } x \in (0, \infty), \quad \phi(0) = 0, \quad |\phi| < \infty.$$
Looking for a solution and applying the BC at $x = 0$ we find that
$$\phi = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x,$$
$$0 = \phi(0) = \implies \phi = \sin\sqrt{\lambda}x \text{ is a solution for any } \lambda > 0.$$

Every positive λ is an eigenvalue with eigenfunction $\phi_\lambda = \sin\sqrt{\lambda}x$. The set of eigenvalues is **continuous**. We cannot use it as a basis, etc. as in the eigenfunction method.[3]

5.2. Non-oscillating solutions: Recall that there were **two main cases** to check for
$$-\phi'' = \lambda\phi \text{ for } x \in (a, b), \quad \phi(a) = \phi(b) = 0,$$
either with oscillating solutions ($\lambda > 0$) or exponentials ($\lambda < 0$), plus a third case at $\lambda = 0$.

In the 'non-oscillating' case, $\lambda < 0$, the basis solutions are monotonic. There is no hope of finding an infinite set of eigenfunctions here because they do not oscillate (and here, there are no eigenfunctions at all). However, there can be **a finite number of eigenfunctions** in this 'non-oscillating' case (we'll see examples later).

5.3. Identifying the cases. The cases are typically separated by $\lambda = 0$, but not always. The breakpoint between oscillating/non-oscillating solutions depends on the ODE, e.g.
$$\phi'' + \phi = -\lambda\phi \implies \text{char. poly. } p(r) = r^2 + r + \lambda$$
$$\implies r = \frac{1}{2}(-1 \pm \sqrt{1 - 4\lambda}) \implies \text{oscillating solutions iff } \lambda > 1/4$$
For another simple but useful example, suppose
$$L\phi = -\phi'' + \gamma\phi, \quad \text{...BCs...}$$
This is a shifted version of the usual $-\phi''$ operator:
$$L\phi = \tilde{L}\phi + \gamma\phi, \qquad \tilde{L}\phi := -\phi''.$$
If $\tilde{\phi}_n$ and $\tilde{\lambda}_n$ are eigenfunctions/values of $\tilde{L}$ then
$$L\phi_n = \tilde{L}\phi_n + \gamma\phi_n = (\tilde{\lambda}_n + \gamma)\phi_n.$$
It follows that the eigenfunctions for L are the same as for $\tilde{L}$, with shifted eigenvalues
$$\lambda_n = \tilde{\lambda}_n + \gamma.$$
That is, Ł is the original operator shifted by a multiple γ of the identity. This means that shifted eigenvalue problems can be solved quickly using standard results e.g.
$$-\phi'' + 6\phi = \lambda\phi, \quad \phi(0) = \phi(\pi) = 0 \implies \phi_n = \sin nx, \quad \lambda_n = n^2 + 6, \quad n \geq 1.$$

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[3]There is a way around this, by using an **integral transform** (see Fourier/Laplace transform).

5.4. **Rayleigh quotient.** A useful trick for showing that eigenvalues are positive. Again, consider as an example the eigenvalue problem

$$-\phi'' = \lambda\phi \text{ for } x \in (a,b), \quad \phi(a) = \phi(b) = 0.$$

Suppose ϕ is an eigenfunction with eigenvalue λ. The goal is to integrate and manipulate the equation to get only squared terms like $\int \phi^2 \, dx$ whose sign is known.

To start, multiply by ϕ and integrate to get

$$-\int_a^b \phi\phi'' \, dx = \lambda \int_a^b \phi^2 \, dx. \tag{5.6}$$

The integral on the left is of unknown sign, so integrate by parts once:

$$\int_a^b \phi\phi'' \, dx = \phi\phi' \Big|_a^b - \int_a^b (\phi')^2 \, dx.$$

Plugging this into (5.6) gives

$$\int_a^b (\phi')^2 \, dx - \phi\phi' \Big|_a^b = \lambda \int_a^b \phi^2 \, dx.$$

We may now 'solve' for λ in terms of ϕ to get the **Rayleigh quotient**

$$\lambda = \frac{-\phi\phi' \Big|_a^b + \int_a^b (\phi')^2 \, dx}{\int_a^b \phi^2 \, dx}. \tag{5.7}$$

Note that ϕ is non-trivial so the denominator is non-zero.

Now return to the example eigenvalue problem with boundary conditions $\phi(a) = \phi(b) = 0$. We claim that the eigenvalues are **strictly positive**. Observe that

$$\lambda = \frac{\int_a^b (\phi')^2 \, dx}{\int_a^b \phi^2 \, dx} \geq 0.$$

Moreover, since $\int_a^b f^2 \, dx = 0$ implies $f(x) = 0$ for all[4] $x \in [a,b]$,

$$\lambda = 0 \iff \phi'(x) = 0 \iff \phi \text{ is constant.}$$

But $\phi(a) = 0$ so this further means that $\phi = 0$. But ϕ is not trivial, so λ cannot be zero.

Summary (Rayleigh quotient): A method to show that eigenvalues are **non-negative** ($\lambda \geq 0$) or **strictly positive** ($\lambda > 0$) without knowing the eigenfunction ϕ. For

$$-\phi'' = \lambda\phi, \quad x \in (a,b), \quad \text{...BCs...,}$$

multiply by ϕ and integrate over (a,b) and **integrate by parts once** to get

$$\lambda \int_a^b \phi^2 \, dx = -\phi\phi' \Big|_a^b + \int_a^b (\phi')^2 \, dx. \tag{5.8}$$

If the boundary terms are ≥ 0, then it follows that $\lambda \geq 0$. To check $\lambda = 0$, use the fact that $\int f^2 \, dx = 0 \implies f = 0$ and see if a solution ϕ can still exist.

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[4]Technically, for almost all - it can be non-zero at isolated points, but the distinction is not important.