MATH 356 LECTURE NOTES
FOURIER SERIES, CONVERGENCE IN L^2

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Topics covered

- Fourier series
  - The main theorem
  - Periodic functions and extensions
  - Examples, computational tricks
  - Sine and cosine series
  - Connection to PDEs
- Convergence of Fourier series
  - Convergence (in L^2)
  - Pointwise and uniform convergence
  - Oscillations at discontinuities (Gibbs' phenomenon)
- Additional topics
  - Decay of coefficients
  - Details of Gibbs' phenomenon

1. Fourier series

Before returning to PDEs, we explore a particular orthogonal basis in depth - the **Fourier series**. This theory has deep implications in mathematics and physics, and is one of the cornerstones of applied mathematics (not just a tool for solving PDEs!). Some results here must be taken on faith, as their proof requires analysis beyond the scope of the course.

This will also provide some exposure to the bases that we will use to solve PDEs later.

1.1. **L^2 as a space of periodic functions.** Here, we primarily work in the L^2 space

$$L^2[-\pi,\ell] = \{f : [-\ell,\ell] \to \mathbb{R} : \int_{-\ell}^{\ell} |f(x)|^2\,dx < \infty\}. \tag{1}$$

This space can be identified with a different space (same elements, **very different interpretation**). Recall that a function is **periodic (with period T)** if

$$f(x+T) = f(x) \text{ for all } x.$$

Every 2ℓ periodic function defined for all $x \in \mathbb{R}$ can be identified with an element of $L^2[-\ell,\ell]$ by restricting to the interval $[-\ell,\ell]$. Nothing is lost since the function repeats after one period.

We may therefore view $L^2([-\ell,\ell])$ as the space of 2ℓ-**periodic** functions defined on all of $\mathbb{R}$.

To go from $f : [-\ell, \ell] \to \mathbb{R}$ to a periodic function on $\mathbb{R}$, we define the 2ℓ-**periodic extension** to be the function given by f on $[-\ell, \ell]$ and

$$f(x + 2\ell) = f(x) \text{ for all } x$$

which defines f on all of $\mathbb{R}$.

Important note (endpoints): If the values at the endpoints do not match, then this just means there is a discontinuity at that point (see examples below) and we assign some value at the discontinuity. Technically, the value at exactly the endpoint is ambiguous and the periodic extension may not agree with f at such points, but that does not matter.

In terms of maps between spaces, we have

$$L^2[-\ell, \ell] \underset{\text{extend to } \mathbb{R}}{\overset{\text{restrict to } [-\ell,\ell]}{\rightleftarrows}} \{f : \mathbb{R} \to \mathbb{R},\ 2\ell\text{-periodic}\}.$$

For example the figure below shows the periodic extensions for

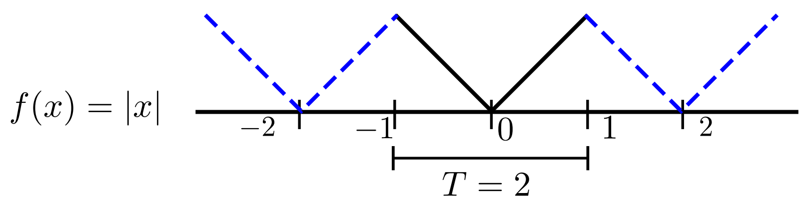
$$f(x) = |x|, \quad x \in [-1, 1] \qquad (2\text{-periodic})$$

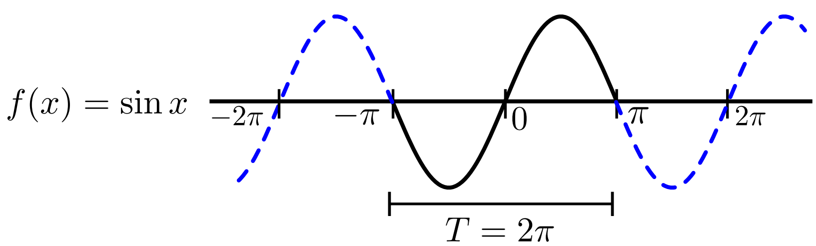
and

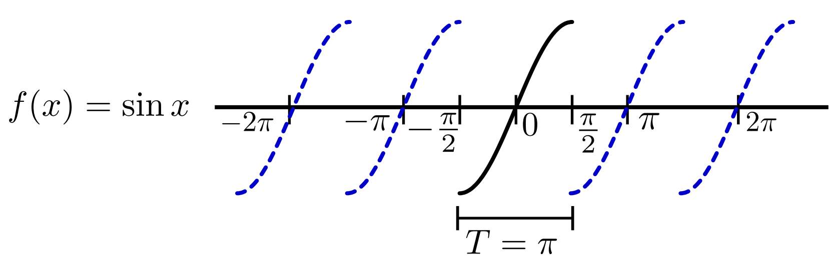
$$f(x) = \sin x, \quad x \in [-\pi, \pi] \qquad (2\pi\text{-periodic})$$

and

$$f(x) = \sin x, \quad x \in [-\pi/2, \pi/2] \qquad (\pi\text{-periodic}).$$







Note that the last two examples show that the periodic extension of a function **will depend on the period specified.** The π-periodic extension of $\sin x$ from $[-\pi/2, \pi/2]$ to $\mathbb{R}$ is different than the periodic extension from $[-\pi, \pi]$ to $\mathbb{R}$.

Theorem (Fourier basis): The set of functions

$$\frac{1}{2} \text{ and } \sin\frac{n\pi x}{\ell},\ \cos\frac{n\pi x}{\ell} \text{ for } n = 1, 2, \cdots \tag{2}$$

is an **orthogonal basis** for the space $L^2[-\ell,\ell]$. That is, every function $f \in L^2[-\ell,\ell]$ has a unique representation (the **Fourier series**)

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \tag{3}$$

with equality in the sense described in subsection 1.3. A single term in the sum, $a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$ is sometimes called the n-th **Fourier mode** of the function. Note that the $\frac{1}{2}$ could be any constant; the value $1/2$ is a convention.

Interpretation: Every periodic function with period T can be decomposed into a sum of sines and cosines whose frequencies are integer multiples of the 'fundamental' frequency $1/T$ (in period/time).

1.2. **Fourier series: the main result.** Since the Fourier series for f on $[-\ell,\ell]$ is 2ℓ-periodic, we can think of (2) as a basis for 2ℓ-periodic functions on $\mathbb{R}$. Often, however, we really only need it to represent a function on some interval, and the periodic extension and periodicity of the series is not needed.

For brevity, let $\phi_0 = 1/2$ and

$$\phi_n = \cos\frac{n\pi x}{\ell}, \quad \psi_n = \sin\frac{n\pi x}{\ell}.$$

Explicitly, the orthogonality relations are

$$\int_{-\ell}^{\ell} \phi_m \psi_n \, dx = 0, \qquad \text{for all } m, n, \tag{4}$$

$$\int_{-\ell}^{\ell} \phi_m \phi_n \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \text{ and } m \neq 0, \end{cases} \tag{5}$$

$$\int_{-\ell}^{\ell} \psi_m \psi_n \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \end{cases} \tag{6}$$

along with the integral for $m = n = 0$.

Because the basis is orthogonal, it is straightforward to compute the coefficients, e.g.

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

The results are given below.

Computing the Fourier series: The coefficients of the Fourier series (3) are given by

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx \tag{7}$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \tag{8}$$

for $n \geq 1$, and

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx.$$

Note that the formula (7) works for $n = 0$ as well.

1.3. **Partial sums and convergence.** The N-th **partial sum** of the Fourier series is the (finite) sum

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n\cos\frac{n\pi x}{\ell}. + b_n\sin\frac{n\pi x}{\ell}.\right)$$

which consists of the Fourier modes up to N.

This partial sum is an approximation to $f(x)$. In fact, it is often a *very good* approximation. We use this to precisely define the equality in (2):

Convergence of Fourier series (mean square): Let $f \in L^2[-\ell,\ell]$. Then the partial sums $S_n(x)$ of its Fourier series converge to f in the L^2 norm; that is,

$$\lim_{N\to\infty}\|f - S_N(x)\|_2 = 0.$$

Explicitly, the 'mean square' (or L^2) distance between the S_N and f goes to zero:

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|S_N(x) - f(x)|^2\,dx = 0. \tag{9}$$

If $f \in L^2[-\pi,\pi]$ we simply write that it is 'equal' to its Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

where the equality is meant in the sense of (9).

The theorem is important because it says that a nice function $f(x)$ on $[-\pi,\pi]$ (or a periodic function) can be approximated by its first N Fourier modes. Adding more modes (more terms to the sum) improves the approximation. In practice, functions are often surprisingly well-approximated by even just a few modes, as we will see. There are, however, some exceptions.

It is critical to note that the convergence result (9) **does not mean** that plugging in a specific value of x gives an equality. That is, it is not necessarily true that

$$\lim_{n\to\infty} S_n(x) = f(x)$$

for all $x \in [-\ell, \ell]$. The examples below illustrate this; technical details are in section 3.

2. Computing Fourier series

Here we compute some Fourier series to illustrate a few useful computational tricks and to illustrate why convergence of Fourier series can be subtle. Because the integral is over a symmetric interval, some symmetry can be exploited to simplify calculations.

2.1. **Even/odd functions:** A function $f(x)$ is called **odd** if

$$f(x) = -f(-x) \text{ for all } x$$

and **even** if

$$f(x) = f(-x) \text{ for all } x.$$

Due to the odd/even symmetry, integrals over intervals symmetric around zero are nice:

$$\text{if } f \text{ is odd, } \int_{-\ell}^{\ell} f(x)\,dx = 0,$$

$$\text{if } f \text{ is even, } \int_{-\ell}^{\ell} f(x)\,dx = 2\int_0^{\ell} f(x)\,dx.$$

Products of even/odd functions are even or odd (hence the name):

$$\text{odd} \cdot \text{odd} = \text{even}, \qquad \text{odd} \cdot \text{even} = \text{odd}.$$

Some common even/odd functions (m is an integer):

$$\text{odd: } x^{2m+1}, \;\; \sin kx, \cdots$$
$$\text{even: } x^{2m}, \quad \cos kx, \cdots$$

As an example,

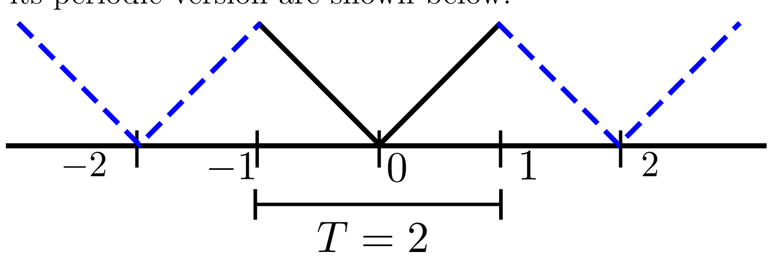
$$\int_{-1}^{1} x^6 \sin 2x + 3x^2\,dx = \int_0^1 3x^2 dx = x^3\Big|_0^1 = 1.$$

For the first term: since x^6 is even, $\sin 2x$ is odd so $x^6 \sin 2x$ is odd.

2.2. **Triangle wave.** Define a function $f \in L^2[-1,1]$ as

$$f(x) = |x| \text{ for } x \in [-1, 1].$$

The function and its periodic version are shown below:



To compute the Fourier series, use (4)-(6) with $\ell = 1$. First, observe that $f(x)$ is an even function, so

$$f(x)\cos n\pi x \text{ is an even function}, \quad f(x)\sin n\pi x \text{ is an odd function} \tag{10}$$

for all n (note that the product of an odd and even function is odd).

For the cosine coefficients, we have

$$a_0 = \int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = 1,$$

and for $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(n\pi x)\,dx \\
&= 2\int_0^1 x\cos(n\pi x)\,dx && \text{(since the integrand is even)} \\
&= \left[\frac{2}{n\pi}x\sin(n\pi x) + \frac{2}{n^2\pi^2}\cos(n\pi x)\right]\Bigg|_0^1 \\
&= \frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 && \text{(since } \sin(n\pi) = 0 \text{ for all n)} \\
&= \frac{2}{n^2\pi^2}((-1)^n - 1).
\end{aligned}$$

Thus the cosine coefficients are

$$a_0 = 1, \qquad a_n = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for odd } n \\ 0 & \text{for even } n > 0 \end{cases}.$$

For the sine coefficients, the integrand is odd due to (10), so

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(n\pi x)\,dx = 0 \text{ for all } n.$$

The Fourier series for f is therefore

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\pi x).$$

The first few partial sums $S_N(x)$ (with modes up to N) are

$$S_1 = \frac{1}{2} - \frac{4}{\pi^2}\cos\pi x, \quad S_3 = \frac{1}{2} - \frac{4}{\pi^2}(\cos\pi x + \frac{1}{9}\cos 3\pi x), \cdots$$

A plot shows that agreement is quite good, even with only a few terms (Figure 1). The error is worst at the peaks of the function, where it has a sharp corner.

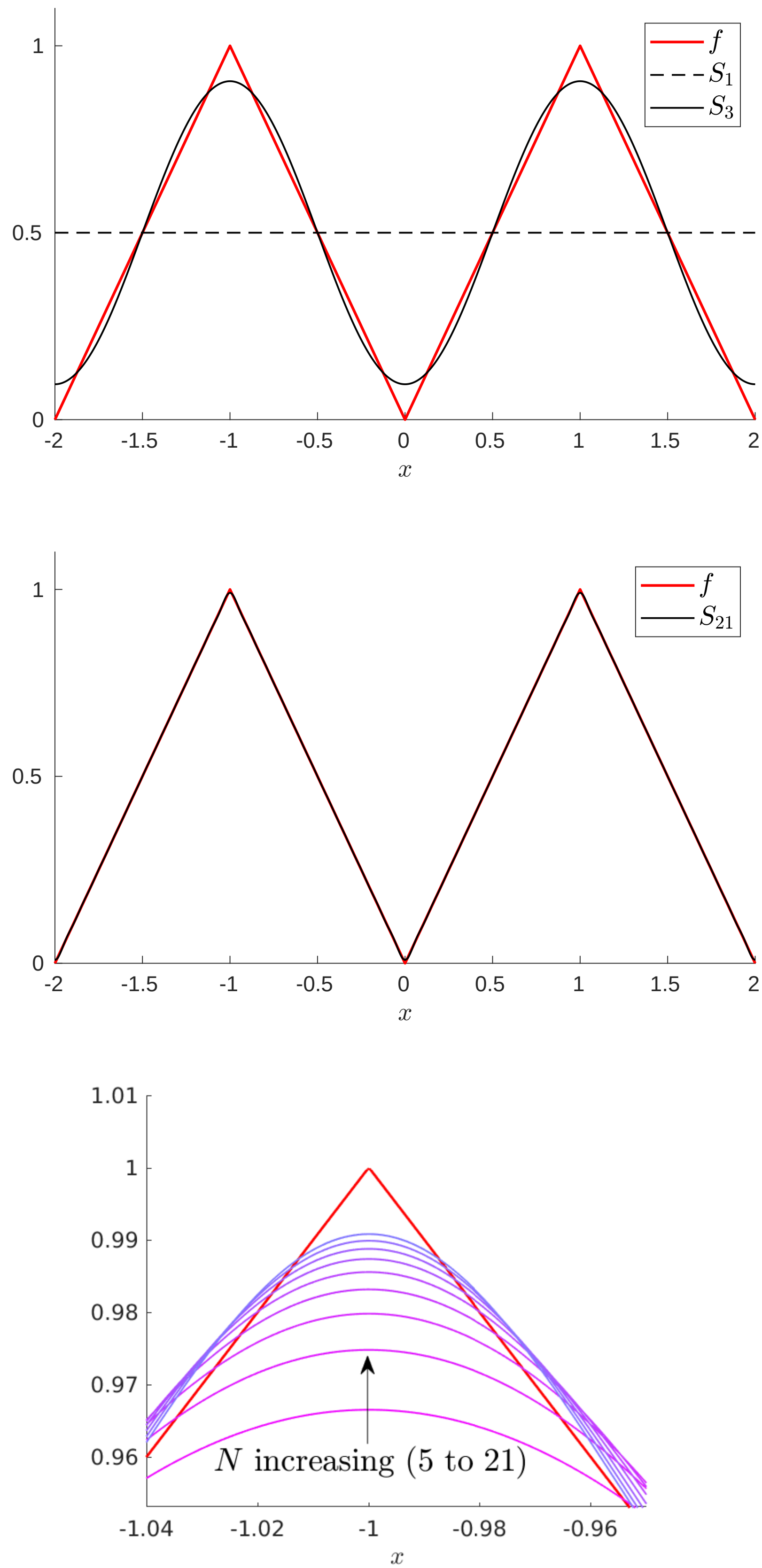
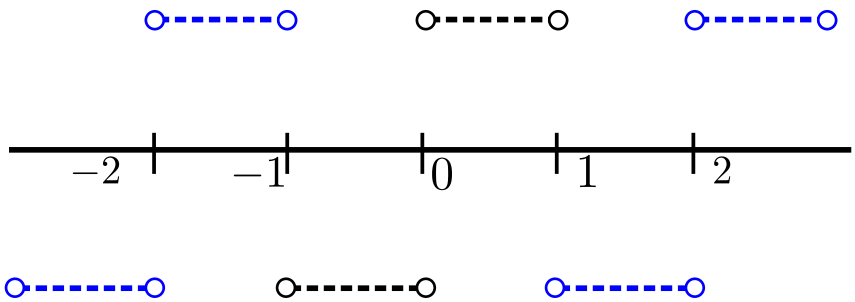


FIGURE 1. Partial sums S_1, S_3 and S_{21} for the triangle wave. Zoomed in plot shows the convergence at a peak of the triangle.

2.3. **Square wave.** Let

$$f(x) = \begin{cases} -1 & -1 \leq x < 0 \\ 1 & 0 < x \leq 1 \end{cases}$$

and $f(x) = f(x+2)$ when $x \notin [-1, 1]$ as shown below:



Note that $f(x)$ is an **odd function** so

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx = 0 \text{ for all } n.$$

For the sine coefficients, use the fact that $f(x) \sin n\pi x$ is an even function:

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x) \sin(n\pi x)\, dx \\ &= 2\int_{0}^{1} \sin(n\pi x)\, dx && \text{(since the integrand is even)} \\ &= -\frac{2}{n\pi} \cos(n\pi x)\Big|_0^1 \\ &= \begin{cases} 4/(n\pi) & \text{for odd } n \\ 0 & \text{for even } n \end{cases}. \end{aligned}$$

Thus the Fourier series for $f(x)$ is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi x).$$

The first few partial sums are

$$S_1(x) = \frac{4}{\pi} \sin \pi x, \quad S_3(x) = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3} \sin 3\pi x\right), \cdots .$$

Error: A plot of the approximation (Figure 2) shows that the partial sums converge nicely where f is continuous, but do not perform well at all near the discontinuity. The partial sums tend to oscillate and overshoot the discontinuity by a significant amount. This overshoot - by about 0.18 - is typical at discontinuities, and is called **Gibbs' phenomenon**.

The oscillations suggest we must be careful with the infinite series - the convergence is not so straightforward. In the next sections, we develop the theory in more detail.

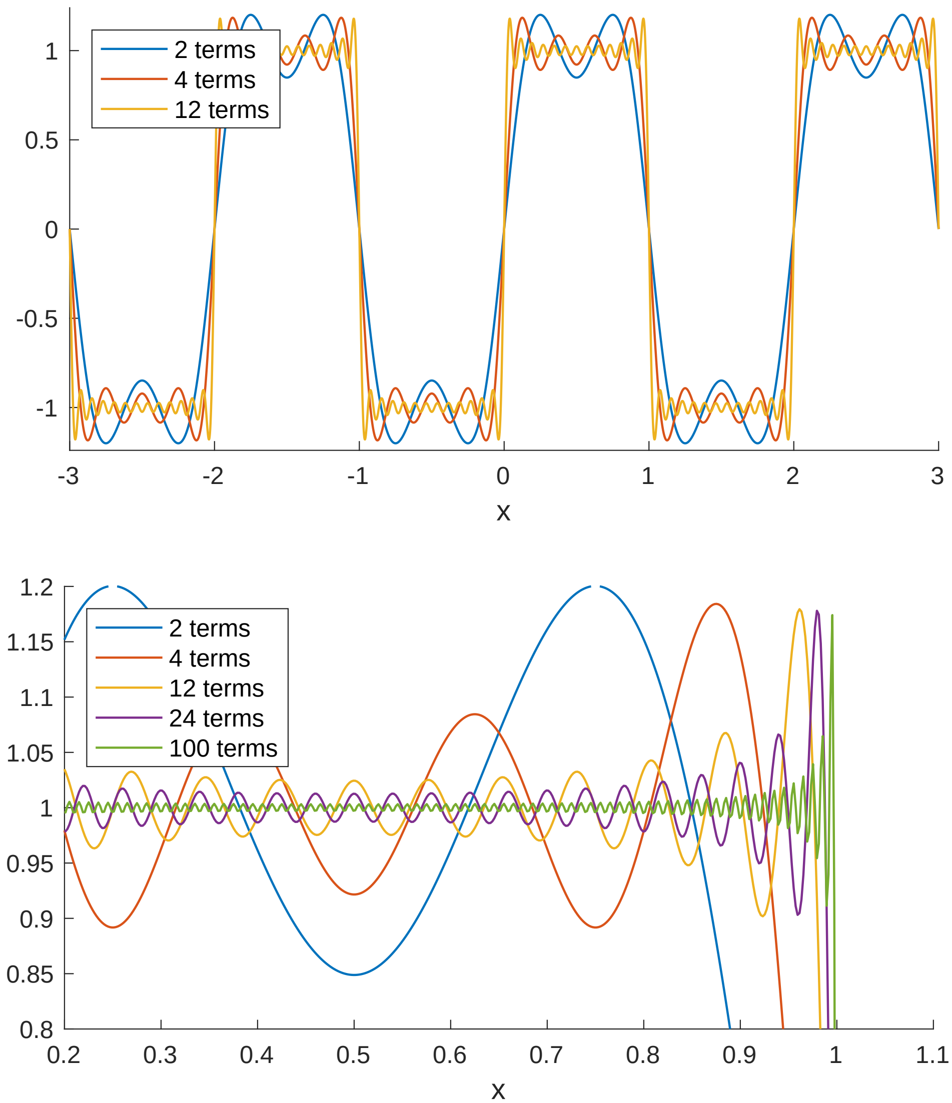


FIGURE 2. Partial sums for the square wave. Zoomed in figure shows the behavior near the corner at $x = 1$. Note that the oscillations around the discontinuity do not decrease in size as $N \to \infty$.

3. Types of convergence

It is worth clarifying the ways the series can converge (or fail to converge), in order to better understand **what equality means** in the Fourier series (3). There are several ways we can measure the error between the partial sum and the function. There are three main notions of convergence that are important here. In this section, we consider a function $f \in L^2[-\ell, \ell]$ and its Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

The N-th partial sum is defined to be the sum of terms up to $n = N$:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right).$$

Convergence (definitions): Let f_n be the sequence of functions in $L^2[-\ell, \ell]$.

The sequence is said to converge **in norm** (or 'in L^2') to a limit f if

$$\|f_n - f\|_2 \to 0 \qquad \text{as } n \to \infty$$

That is,

$$\int_{-\ell}^{\ell} |f_n(x) - f(x)|^2 \, dx \to 0 \text{ as } n \to \infty.$$

The sequence converges **pointwise** to f if

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in [-\ell, \ell].$$

The sequence converges **uniformly** to f if

$$\max_{x\in[-\ell,\ell]} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

L^2 **convergence** means that the mean-square error goes to zero; the weighted average of the area between the partial sums and the function goes to zero. However, it does not require convergence at each point (for instance, the square wave in the previous section).

Pointwise convergence is simpler: it says that at each point x, the value of the partial sums at x will converge to the value of $f(x)$. It does not, however, require that the partial sums converge at the same rate at each x. It could be that at some points, $S_N(x) \to f(x)$ quickly, but at other points, it converges (arbitrarily) slowly (see box below).

Uniform convergence says that the maximum error[1] between $S_N(x)$ and $f(x)$ decreases to zero as $N \to \infty$.

In general, neither of pointwise or norm convergence implies the other. Uniform convergence is stronger than the other two.

Example (pointwise but not uniform): Consider

$$f_n = x^n \text{ on } [0,1], \quad n = 1, 2, \cdots$$

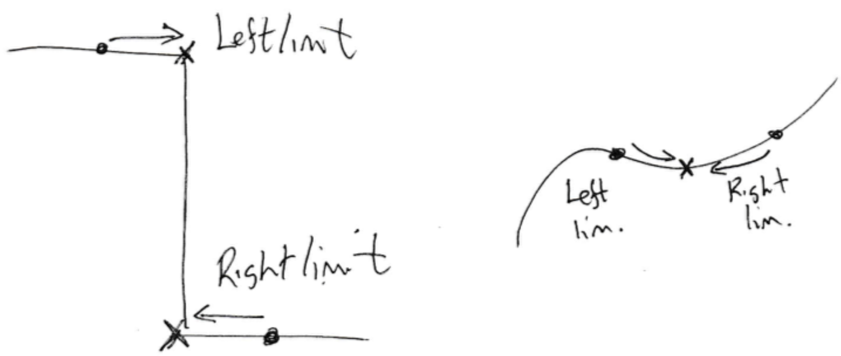
which converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

[1]technically, the 'max' here should be $\sup |f_n(x) - f(x)|$ (the least upper bound of the error) since the maximum may not be achieved.

To see this, note that $f_n(1) = 1$ for all n (so $f_n(1) \to 1$) and
$$\lim_{n\to\infty} x^n = 0 \text{ if } 0 < x < 1.$$
However, for $x < 1$,
$$|f_n(x) - f(x)| = x^n$$
and x^n can be made arbitrarily close to 1 by taking x close enough to 1 (for any n). Thus the maximum error is always 1. The interval where the error is near 1 shrinks in size as n increases, but the max. error never decreases.

3.1. **Pointwise and uniform convergence for the Fourier series.** The Fourier series is defined for functions in L^2, which allows for discontinuities. We will need to be precise about values at discontinuities. For a function with jump discontinuities, define the 'right' and 'left' limits
$$f(x^+) = \lim_{\xi \searrow 0} f(\xi), \quad f(x^-) = \lim_{\xi \nearrow 0} f(\xi).$$



If f is continuous at x then $f(x^+) = f(x^-) = x$. We have the following result (due to Dirichlet in the early 1800s):

Theorem (Pointwise convergence): Let $S_n(x)$ be the n-th partial sum of the Fourier series for a periodic function $f \in L^2[-\ell, \ell]$.

(i) If f (as a periodic function) and f' are continuous, then the partial sums converge to $f(x)$ **uniformly** as $n \to \infty$, i.e.
$$\lim_{n\to\infty} S_n(x) = f(x) \text{ for all } x \in [-\ell, \ell].$$
(ii) If f and f' are continuous except at some jump discontinuities, then
$$\lim_{n\to\infty} S_n(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{1}{2}(f(x_-) + f(x_+)) & \text{if } f \text{ has a jump at } x. \end{cases}$$
That is, the partial sums converge to the average of the left and right limits.

Note that for f to be continuous as a periodic function, it must be true that the endpoint values match, i.e. $f(-\ell) = f(\ell)$. If they do not, then only the second part applies.

For the square wave example, the partial sums converge to the average $(-1+1)/2 = 0$ at $x = \pm\ell$ and $x = 0$. Otherwise, the partial sum $S_n(x)$ at a point x will converge to $f(x)$.

Note that 'f' is piecewise continuous' means that between the discontinuities, f has a continuous derivative[2].

Interpretation: As shown in the sketch, to fit the discontinuity the sines/cosines need to have a maximum at the top and minimum at the bottom of the jump; thus the oscillations will oscillate around the midpoint.

It tends to be true that if there is a discontinuity, the convergence near this jump (excluding the jump itself) is pointwise (as guaranteed by (ii)) but not uniform. That is, there will always be an overshoot of some non-vanishing size, no matter how many terms we add. The oscillations around the discontinuity are discussed in detail in the next section.

4. Examples of convergence

4.1. **Triangle wave.** Consider again the triangle wave

$$f(x) = |x| \text{ for } x \in [-1,1], \qquad f(x) = f(x+2).$$

Note that since $f(-1) = f(1) = 1$, the endpoints match, so the periodic extension will be continuous at these points. Since $f(x)$ is otherwise continuous, we see that f (as a 2-periodic function) is continuous.

Similarly, we have

$$f'(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

and $f'(x)$ undefined at $x = 0, \pm 1$. Thus f' is piecewise continuous.

It follows from the convergence theorem that the partial sums converge **pointwise** to $f(x)$, and moreover they actually converge **uniformly** to $f(x)$.

The situation we observed in Figure 1 agrees with the theorem; The error is largest at the corner, and that error decreases to zero as $N \to \infty$ (albeit slowly).

4.2. **Square wave.** We may now finish discussing convergence the square wave.

Recall that we found the Fourier series for the square wave defined by

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 < x \le 1 \end{cases} \tag{11}$$

[2]The assumptions here are only sufficient; the theorem can be relaxed somewhat but the analysis becomes much more difficult. Fortunately, most functions in practice will be nice between discontinuities.

and $f(x) = f(x+2)$ when $x \notin [-1,1]$. Let us consider $f(x)$ on the interval $[-1,1]$. The Fourier series for $f(x)$ is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin((2n-1)\pi x).$$

As a periodic function in $L^2[-1,1]$, $f(x)$ is continuous except for discontinuities at $x = 0$ and $x = \pm 1$. Note that $f(-1) = -1$ and $f(1) = 1$; the endpoints do not match, so the periodic version has discontinuities at $\pm 1, \pm 3, \cdots$ and so on.

Pointwise convergence: For the purposes of the convergence theorem, f defined on $[-1,1]$ has discontinuities at the endpoints, and

$$f(1^-) = -1, \quad f(1^+) = -1.$$

We also have

$$f(0^-) = 1, \quad f(0^+) = 1.$$

It follows that the partial sums converge to $f(x)$ when $x \neq -1, 0, 1$ and converge to 0 at all the discontinuities (the average is always $\frac{1}{2}(-1+1) = 0$). Define

$$\tilde{f}(x) = \frac{1}{2}(f(x^-) + f(x^+)) = \begin{cases} f(x) & \text{if } x \neq 0, \pm 1 \\ 0 & x = 0, \pm 1 \end{cases}.$$

Then the convergence theorem says that

$$\tilde{f}(x) = \lim_{N\to\infty} S_N(x) \text{ for } x \in [-1,1].$$

In particular, it does not matter how $f(x)$ in (11) is defined at the discontinuities; the Fourier series will converge to $\tilde{f}(x)$ regardless of the values chosen for $f(\pm 1)$ and $f(0)$.

Remark: Note that in this case, the theorem is not needed to see what happens at the discontinuities since

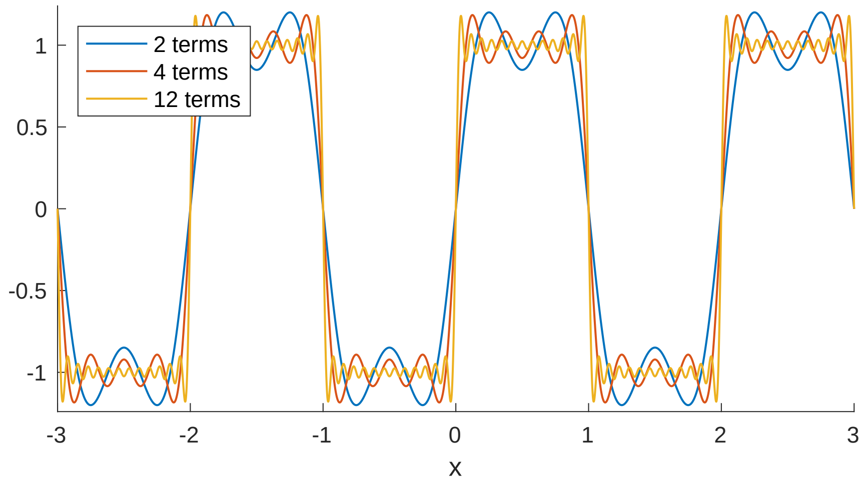
$$S_N(0) = S_N(\pm 1) = 0 \text{ for all } N$$

as all the terms are zero individually.

Uniform convergence: However, $f(x)$ is not continuous, so we cannot conclude that the convergence is uniform. Indeed, it is not, as we have seen by direct inspection. The overshoot at the discontinuities never goes away, and it is true that

$$\max_{x\in[-1,1]} |f(x) - S_m(x)| \approx 0.18 \text{ as } m \to \infty.$$

Proving this requires some work; see section 7.



The partial sums tend to oscillate and overshoot the discontinuity by a significant amount, no matter how large we make N. The persistent overshoot here is called **Gibbs' phenomenon**, and is how Fourier series generically behave at discontinuities.

Gibbs' phenomenon (summary): If $f(x)$ has a discontinuity then the partial sums $S_N(x)$ of the Fourier series will over-shoot the true value of $f(x)$ at the top part of the discontinuity and undershoot the bottom part. The amount of the overshoot approaches a **constant** as $N \to \infty$ and the overshoot is roughly 9% of the jump height on each end.

Why not uniform convergence? Since $f \in L^2[-1,1]$, we have that

$$\|f - S_N\|^2 = \int_{-1}^{1} |f(x) - S_N(x)|^2 \, dx \to 0 \text{ as } N \to \infty.$$

Thus, we have that

- S_N converges in norm to f
- S_N converges pointwise to $\tilde{f}$ (to f except at discontinuities)
- S_N does not converge uniformly to f: the max. error is always around 0.18.

How can it be that the series converges at (almost all) points, and yet the maximum error never decreases?

Why in norm, but not uniform: The approximation gets better away from the discontinuities (the 'good' region) but always overshoots near the discontinuities (the 'bad' region); see figure below. Because of the integral over $[-1,1]$ the mean square error, very roughly, looks like

$$\|f - S_N\|^2 \sim \text{error in good region} + (\text{height of overshoot}) \times (\text{width of bad region}).$$

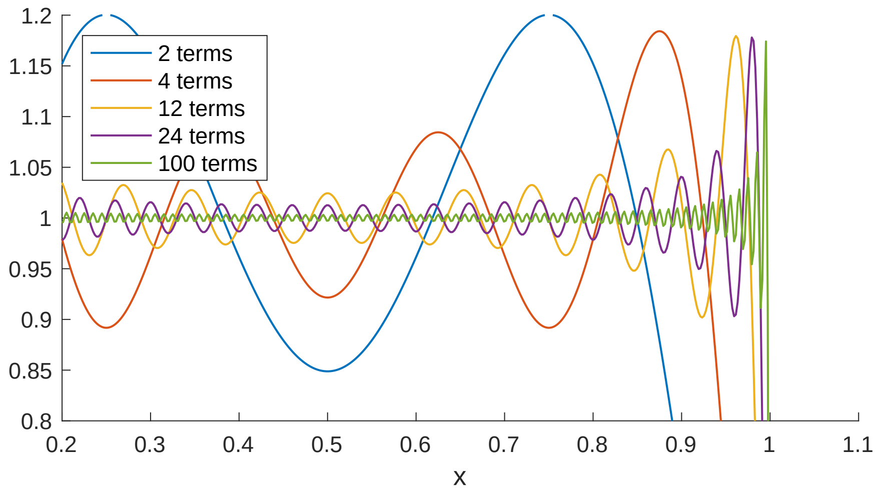
As $N \to \infty$, we observe that the width of the bad region goes to zero while hte height stays the same, so the second term goes to zero. In the good region, the oscillations around ± 1 shrink, and the error decreases nicely at each point so the first term also goes to zero.

Why pointwise, but not uniform: Suppose $x_0 \in (0,1)$. Fix this value and consider

$$\lim_{N\to\infty} S_N(x_0).$$

Observe from the figure that the 'bad' region where the error is large gets compressed closer and closer to $x = 1$. For large enough N, it is confined to a small interval around 1, so x_0 will eventually be outside the bad region.

Outside the bad region, the oscillations decrease in magnitude, and convergence is nice.



5. Sine and cosine series

Suppose $f(x)$ is defined on $[0, \ell]$. Here we consider the problem of finding a Fourier series that is 2ℓ periodic and is equal to f on $[0, \ell]$. We can 'fill in' the other half $[-\ell, 0]$ with an arbitrary function, then extend to a periodic function $\mathbb{R}$ using $f(x) = f(x + 2\ell)$.

Any such extension to $[-\ell, \ell]$ has a (unique) Fourier series, which will be equal to $f(x)$ on $[0, \ell]$. The freedom to choose a series (by picking its values on $[-\ell, 0]$) will be useful later in solving boundary value problems. We will make use of the symmetry of odd and even functions as defined in section 2. First, as seen in section 2, note that if $f(x)$ is an odd function then $f(x) \cos n\pi x/\ell$ is odd for all n so

$$a_n = \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx = 0 \text{ for all } n.$$

Since $f(x) \sin n\pi x/\ell$ is even,

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx.$$

A similar argument holds for even functions, leading to a useful rule:

Fourier series for odd/even functions: If $f \in L^2[-\ell, \ell]$ is an even function then

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

where

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx.$$

If $f \in L^2[-\ell, \ell]$ is an odd function then

$$f = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx.$$

5.1. **Odd/even periodic extensions.**

For $f(x)$ defined on $[0, \ell]$, the two important extensions are the ones that have odd/even symmetry:

The **even extension** of $f(x)$ defined on $[0, \ell]$ to the new interval $[-\ell, \ell]$ is

$$g(x) = \begin{cases} f(x) & 0 \le x < \ell \\ f(-x) & -\ell < x \le 0. \end{cases}$$

That is, we fill in the $[-\ell, 0]$ part by reflecting $f(x)$ across the y-axis.

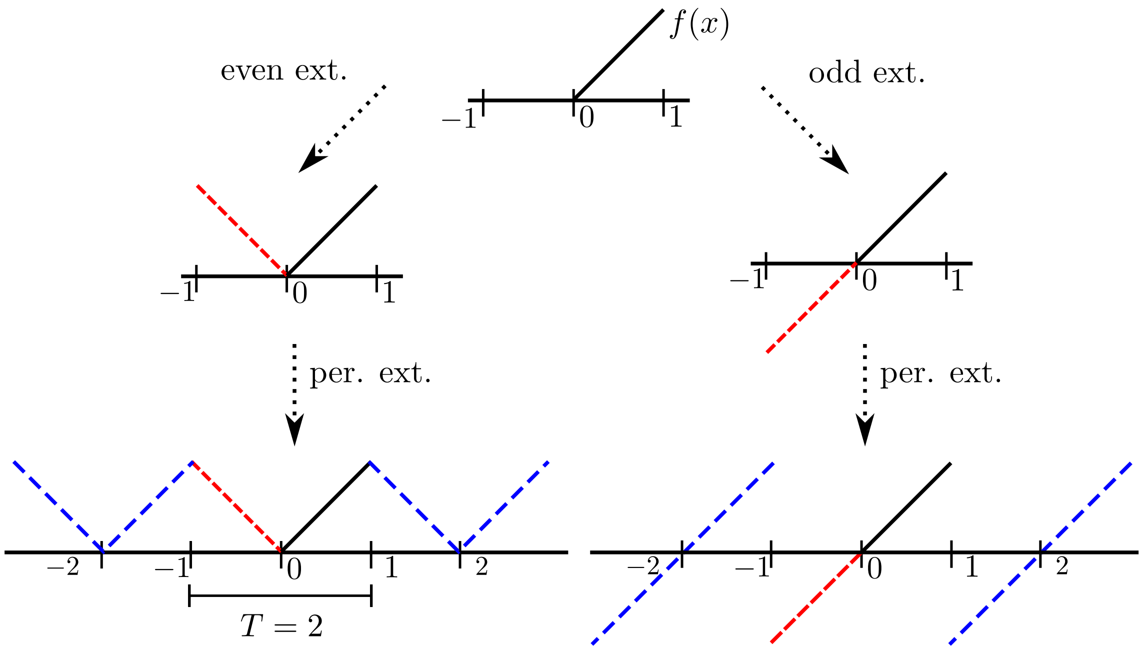
The **odd extension** of $f(x)$ to $[-\ell, \ell]$ is

$$g(x) = \begin{cases} f(x) & 0 < x < \ell \\ 0 & x = 0 \\ -f(-x) & -\ell < x < 0. \end{cases}$$

It is easy to see that the even extension is always an even function and the odd extension is always an odd function. The even/odd extensions of the function

$$f(x) = x \qquad \text{for } x \in [0, 1]$$

to $[-1, 1]$ and the corresponding periodic extensions (with period 2) are shown below.



Remark (other extensions): For a function $f \in L^2[0, \ell]$, there are many choices for extensions to a 2ℓ-periodic function. We could, in in principle, fill in whatever values we want in $[-\ell, 0]$. The even/odd extensions are discussed here because they have nice/useful properties due to the symmetries.

Other extensions are useful too, but are more specific so they will not be discussed.

5.2. **Fourier sine and cosine series.** Now let's consider a function $f(x)$ on $[0, \ell]$. Suppose we want to write f in terms of a Fourier series (using the basis for $[-\ell, \ell]$). This can be done by extending f to $[-\ell, \ell]$ in whatever way, then finding the series for the extension. The two notable ones are:

The **Fourier sine series** for $f(x)$ is the Fourier series of the **odd** extension of $f(x)$ to $[-\ell, \ell]$. By the previous discussion, all the cosine coefficients will vanish (hence the name).

The **Fourier cosine series** for $f(x)$ is the Fourier series of the **even** extension of $f(x)$ to $[-\ell, \ell]$. All the sine coefficients vanish.

Note that the above means that a function on $[0, \ell]$ (without any other context) has a Fourier sine series and a Fourier cosine series,**both of which equal f (in the appropriate sense) in** $(0, \ell)$. However, they **do not** agree on $[-\ell, 0]$. That is, we can find representations

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/\ell)$$

and

$$f = \qquad \sum_{n=1}^{\infty} b_n \sin(n\pi x/\ell)$$

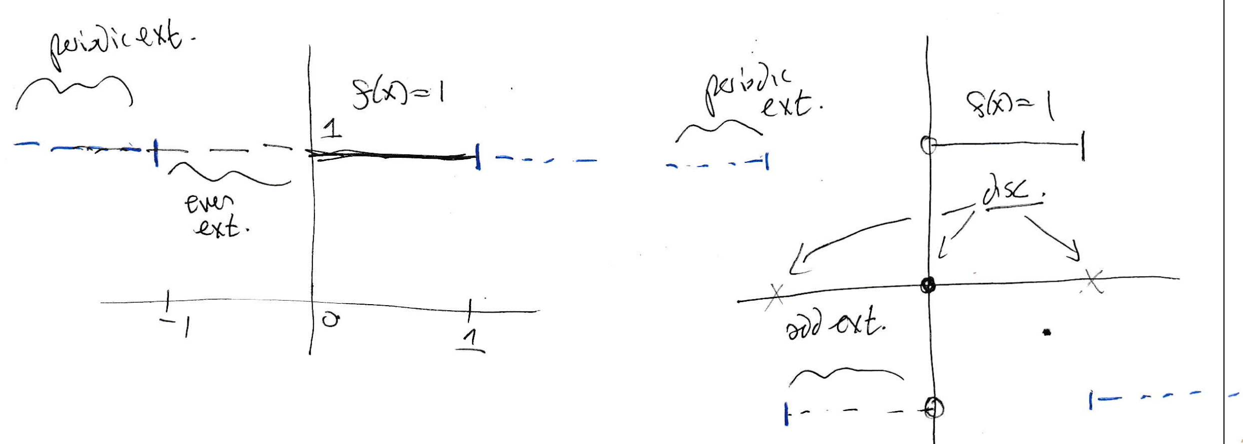
where both series converge to the same function in $(0, \ell)$.

It is important to note that the extension of f to an even/odd periodic function may introduce a discontinuity at $x = 0$ or $x = \ell$ (see example below)!

An example of sine/cosine series: Consider the function

$$f(x) = 1, \qquad x \text{ in } [0, \pi].$$

Suppose we wish to find a Fourier series representation for $f(x)$ with period 2π. Because only half the function values are specified, we have freedom to fill in the other half - so there are many possible choices for an approximation. The periodic even/odd extensions (with period 2) are shown below:



The cosine series with period 2π for $f(x)$ is the Fourier series for

$$f(x) = 1, \qquad x \text{ in } [-\pi, \pi].$$

This series is quite easy to find, because $f(x) = 1$ is orthogonal to all the basis functions except 1. Thus all the Fourier coefficients are zero except a_0, and the series is just

$$f(x) = 1,$$

which is (trivially) the series for the 'periodic' function $f(x) = 1$ defind for $x \in \mathbb{R}$.

On the other hand, the sine series with period 2π is the Fourier series for

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x = 0 \\ -1 & -1 < x < 0 \end{cases}, \qquad x \in [-\pi, \pi].$$

This series (which was computed earlier) is

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n-1} \sin((2n-1)x).$$

The series converges to $f(x)$ for $x \in (-\pi, \pi)$ and to 0 (the average of -1 and 1 when $x = \pm\pi$. Note that the sine/cosine series agree on $(0, \pi)$ but **not** at $x = 0$ or $x = \pi$.

There are other possible extensions. If extended to be zero for $[-1, 0]$, i.e.

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

then the Fourier series coefficients are given by $a_0 = 1$ and

$$a_n = \int_0^1 \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x)\Big|_0^1 = 0,$$

and

$$b_n = \int_0^1 \sin(n\pi x)\, dx = -\frac{1}{n\pi}(\cos(n\pi) - 1) = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

so the Fourier series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)x).$$

It converges to 1 when $x \in (0, \pi)$ like the sine/cosine series and $1/2$ when $x = 0$ or $x = \pi$. Note that this series is neither a sine or cosine series (it has non-zero a_n and b_n terms).

Remark: Notice that $f(x) - 1/2$ is an odd function, so we could have found the Fourier series for $f(x) - 1/2$ knowing in advance that it should contain only sines.

6. Decay of coefficients

The details of the derivation here are not essential; the main point is to gain some intuition for what the Fourier coefficients look like. Consider $L^2[-\pi,\pi]$ and the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{12}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{13}$$

One key question is the following:

How fast do the coefficients a_n, b_n decay with n?

Worst case: It should be plausible that at the very least, we need them to go to zero as $n \to \infty$ for the sum to have any hope of convergence. The fundamental result is the following:[3]

Riemann-Lebesgue lemma : If $f \in L^2[-1,1]$ is continuous then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0.$$

In particular, this means that for the Fourier series of f,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

The rough idea is that the oscillation in $\sin nx$ causes more and more cancellation in the integral as the frequency increases.

Better case: By integrating by parts, we can get a little more. Suppose f' is continuous and f is continuous as a periodic function, i.e. f and f' are continuous in $[-\pi,\pi]$ and

$$f(-\pi) = f(\pi).$$

Then

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} f(x)\sin n\pi x\,dx \\ &= -\frac{1}{n\pi} f(x)\cos n\pi x\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos n\pi x\,dx \\ &= \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos n\pi x\,dx. \end{aligned}$$

But if f' is continuous on $[-\pi,\pi]$ then $|f'|$ has a maximum value, say M, so

$$|b_n| \le \frac{1}{n\pi}\int_{-\pi}^{\pi} M\,dx = \frac{2}{n}.$$

A similar result holds for the a_n's. The process can be continued so long as we have more derivatives of f and they are continuous as periodic functions. The result is:

[3] The assumptions on f can be relaxed somewhat; they are simplified a bit here.

Basic coefficient bounds: If $f \in L^2[-\pi,\pi]$ and its derivatives up to $f^{(k-1)}$ are continuous as periodic functions and $f^{(k)}$ is piecewise continuous then

$$|a_n| \le \frac{C}{n^{k+1}}, \quad |b_n| \le \frac{C}{n^{k+1}}$$

for some constant $C = C(k)$ (independent of n but dependent on k).

The general principle is then that

$$\text{smoother } f \implies \text{faster convergence of its Fourier series.}$$

Informally, we get **one factor of $1/n$ for each derivative** (as a periodic function).

Examples: For the square/triangle examples:

$$\text{square} \implies \text{jump in } f \implies |a_n|,|b_n| \le C/n$$
$$\text{tri.} \implies f \text{ cts.} + \text{jump in } f' \implies |a_n|,|b_n| \le 1/n^2.$$

The rule gives a quick way to determine how fast coefficients decay. Consider the 'sawtooth'

$$f(x) = x, \quad x \in [-1,1], \text{ period } 2.$$

Since $f(-1) \neq f(1)$, this function is not continuous so the decay rate is $1/n$.
Now consider

$$f(x) = x^2, \quad x \in [-1,1], \text{ period } 2.$$

This function is continuous periodic ($f(-1) = f(1)$) but its derivative is not ($f'(-1) \neq f'(1)$) so the decay rate is $1/n^2$.

6.1. **Convergence proof: easy case [extra].** This result provides a simple proof of convergence under stronger conditions than the convergence theorem of section 3.

Simple convergence theorem: if $f \in L^2[-\pi,\pi]$ and f', f'' are both continuous (as periodic functions) then the Fourier series converges uniformly to f, i.e.

$$\max_{x\in[-\pi,\pi]} |f(x) - S_N(x)| \to 0 \text{ as } N \to \infty.$$

Proof. (Sketch) By the bounds on the coefficients, $|a_n|$ and $|b_n|$ are bounded by C/n^2 for some constant C. We can then estimate

$$|S_N(x) - f(x)| = \left| \sum_{n=N+1}^{\infty} (a_n \cos nx + b_n \sin nx) \right| \le C \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

But we know from calculus that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N}.$$

It follows that for all $x \in [-\pi,\pi]$,

$$|S_N(x) - f(x)| \le \frac{C}{N}.$$

Thus the maximum error is C/N which goes to zero as $N \to \infty$. ∎

7. Extra: Gibbs' phenomenon, details

Let's look at again the Fourier series for the square wave f_{sq} and triangle wave f_{tr} from the previous examples:

$$f_{sq}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin((2n-1)\pi x).$$

$$f_{tr}(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos((2n-1)\pi x).$$

For the triangle wave, we know from the convergence theorems that the partial sums converge uniformly to f. Fix a point x; then notice that the absolute value of the sum is bounded by

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} < \infty.$$

So the coefficients decrease like $1/n^2$, which is fast enough that it doesn't matter what the values of the cosines are. On the other hand, for the square wave, the coefficients only decrease like $1/n$, and we know that

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \infty.$$

Thus for the partial sums to converge to the square wave, there must be some cancellation due to the sines to have convergence, e.g. something like

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

which is convergent. But it is possible for the sines to 'align' so that the terms in the partial sum don't cancel, and instead accumulate - which will lead to oscillations that never go away.

We can be more precise about the mysterious 9% with a little work: we find the location of the peak closest to the discontinuity and compute its height. Consider Gibb's phenomenon at $x = 0$ for $s_{sq}(x)$. The partial sum

$$S_N(x) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{1}{2n-1}\sin((2n-1)\pi x)$$

can be shown to have a critical point at $x = \pi/2N$ (a point close to $x = 0$; this is the location of the largest 'peak' of the oscillation) and

$$S_N(\pi/2N) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{1}{2n-1}\sin(\frac{(2n-1)\pi}{2N}).$$

Up to a constant, this sum is an approximation to the integral of $F(x) = \sin x/x$ over $[0, \pi]$:

$$\int_0^{\pi} F(x)\,dx \approx \frac{\pi}{N}\sum_{n=1}^{N} F(\frac{(2n-1)\pi}{2N}).$$

(The integral is the area under the curve; estimate using rectangles of width π/N with height equal to the value of F at the midpoint). Precisely, we have

$$S_N(\pi/2N) = \frac{2}{\pi}\frac{\pi}{N}\sum_{n=1}^{N}\frac{1}{(2n-1)\pi/(2N)}\sin(\frac{(2n-1)\pi}{2N}) \approx \frac{2}{\pi}\int_0^{\pi}\frac{\sin x}{x}\,dx \approx 1.18$$

which is 9% times the jump height (which is 2 for f_{sq}) above the correct value of 1. As $N \to \infty$, this overshoot converges to the above value.