MATH 356 LECTURE NOTES
LINEAR ODES AND SYSTEMS:
GENERAL STRUCTURE; SYSTEMS IN $\mathbb{R}^2$

J. WONG (FALL 2019)

Topics covered

- Linear systems (homogeneous)
  - Existence/uniqueness for linear systems
  - Basis for solutions, linear independence
  - Linear constant-coefficient (LCC) systems in $\mathbb{R}^2$
  - Second-order ODEs as systems
  - (The Wronskian and fundamental matrix)

1. Linear systems of ODEs

A **linear system of first-order ODEs** in $\mathbb{R}^n$ (here, 'linear system' for short) for a vector-valued function $\mathbf{x}(t)$ has the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \tag{1.1}$$

where $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ and $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ (i.e. $A(t)$ is an $n \times n$ matrix at each t).[1] Note that the system is of the form $L[\mathbf{x}] = \mathbf{f}$ for the linear operator

$$L[\mathbf{x}] := \mathbf{x}' - A(t)\mathbf{x}. \tag{1.2}$$

Existence/uniqueness theorem: To discuss briefly, the existence theorem can be extended to show that

a unique solutions exists where the coefficients $A(t)$ and $f(t)$ are continuous.

For simplicity, **we omit the issue of the domain in this section**. Results said or implied to hold 'for all t' will hold in the interval I where the A and f are continuous. Also, we assume the hypotheses of the theorem always hold.

We saw earlier that solutions to a homogeneous first-order linear ODE form a vector space of dimension one. Let us now do the same for a homogeneous linear system of ODEs

$$\mathbf{x}' = A(t)\mathbf{x}, \qquad \mathbf{x} : \mathbb{R} \to \mathbb{R}^n. \tag{1.3}$$

One key property is the principle of **superposition**, which says that

$$\mathbf{x}_1(t), \mathbf{x}_2(t) \text{ are solutions to (1.3)} \implies \text{any linear combination is also a solution.}$$

This says that the set of solutions

$$\{\mathbf{x}(t) : \mathbf{x}' = A(t)\mathbf{x}\}$$

[1] Note that the most general linear function of a vector in $\mathbb{R}^n$ is $x \to Ax$ where A is an $n \times n$ matrix, so it follows that the above is the most general form of a linear system.

is a vector space. In practice, it lets us build solutions from linear combinations of others.

1.1. **Linear independence.** To find a basis, we must first define what it means for *solutions* (which are functions) to be linearly independent. The motivation is this: if $\mathbf{x}_1, \cdots \mathbf{x}_n$ are to be a basis for solutions to (1.3), then the set of solutions to the IVP

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

must be spanned by the basis solutions $\{\mathbf{x}_i\}$. That is, there must be a solution to the equation

$$c_1\mathbf{x}_1(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{x}_0$$

for each $\mathbf{x}_0$, i.e. the vectors $\mathbf{x}_1(t_0), \cdots, \mathbf{x}_n(t_0)$ must be linearly independent.

Lemma (Linear independence of solutions) A set of functions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ is called linearly independent if there is a value t_0 such that

$$\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots \mathbf{x}_k(t) \text{ are linearly independent vectors at } t = t_0. \tag{1.4}$$

If the functions are **solutions** to to the homogeneous ODE (1.3) then this is equivalent to the stronger condition that

$$\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots \mathbf{x}_k(t) \text{ are linearly independent vectors for all } t. \tag{1.5}$$

The proof of the equivalence uses uniqueness to 'transport' the linear independence property from t_0 (where it is assumed) to all t (which we want to show).

Proof. Clearly one direction is trivial.

For the other, let $\mathbf{x}_1(t), \cdots, \mathbf{x}_k(t)$ be solutions to $\mathbf{x}' = A(t)\mathbf{x}$ and suppose (1.4) holds. Pick an arbitrary value t_1.; we claim that the vectors $\mathbf{x}_1(t_1), \cdots, \mathbf{x}_k(t_1)$ are linearly independent. Suppose that

$$c_1\mathbf{x}_1(t_1) + \cdots c_k\mathbf{x}_k(t_1) = 0$$

for some scalars $c_1, \cdots, c_k$. We need to show that these scalars are all zero. Define

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t).$$

Observe that $\mathbf{x}(t)$ solves the IVP

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(t_1) = 0.$$

But the zero function also solves the IVP, so by uniqueness, $\mathbf{x}(t) \equiv 0$ (i.e. is zero for all t). But this means that

$$0 = \mathbf{x}(t_0) = c_1\mathbf{x}_1(t_0) + \cdots + c_n\mathbf{x}_k(t_0).$$

At t_0, the vectors in the above are linearly independent, so $c_1 = \cdots = c_n = 0$. □

1.2. **Solution space, basis.** Now we can describe solutions to the homogeneous problem.

Main claim: The homogeneous solution space (the null space of L in (1.2)),

$$\{\mathbf{x}(t) : \mathbf{x}' = A(t)\mathbf{x}\}$$

has dimension n. That is, we can find a set of n linearly independent 'basis solutions' $\mathbf{x}_j(t)$ that span the set of solutions, i.e. the general solution to the ODE is

$$\mathbf{x} = \sum_{j=1}^{n} c_j \mathbf{x}_j(t).$$

1.3. **Basis for solutions: proof.** We return to the promised proof that the basis of n solutions exists. The proof uses the existence theorem to construct n solutions, then show that their span is all solutions.

Structure of the solution space: There are n linearly independent solutions $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ to the homogeneous ODE

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

that form a basis for the solution space.

The proof is useful to see (optional, but worth understanding). It is an application of the existence theorem to generate the n basis solutions.

Proof. First, we use linearly independent initial conditions and existence to produce linearly independent solutions (the basis). Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$ and pick any t_0. By the existence theorem,

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{v}_i \text{ has a unique solution for } i = 1, \cdots, n.$$

By construction, the $\mathbf{x}_i$'s are linearly independent at t_0 so they are linearly independent functions by the lemma. This gives the proposed basis.

Now we show the $\mathbf{x}_i$'s are really a basis for solutions. Let $\mathbf{x}(t)$ be any solution to the ODE; we wish to show it is a linear combination of the $\mathbf{x}_i$'s. To do so, pick a point t_0 and let $\mathbf{x}_0 = \mathbf{x}(t_0)$. Then $\mathbf{x}(t)$ solves the initial value problem

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{1.6}$$

Since $\mathbf{x}_1(t_0), \cdots, \mathbf{x}_n(t_0)$ are linearly independent *vectors*, there are scalars $c_1, \cdots c_n$ such that

$$c_1\mathbf{x}_1(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{x}_0.$$

Now combine them into a solution

$$\mathbf{y}(t) := c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t).$$

Both $\mathbf{x}(t)$ and $\mathbf{y}(t)$ solve the IVP (1.6). By uniqueness, they are equal as functions:

$$\mathbf{x}(t) = \mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \quad \text{for all } t$$

so $\mathbf{x}(t)$ is indeed in the span of the basis solutions. □

So what is the basis? The proof demonstrates that the basis exists but does not construct it. So how do we compute solutions? Short answer: **In general, we can't expect to have explicit solutions**. There are only special cases where the basis solutions can be found, and a few tricks for obtaining solutions from other solutions.

2. Systems in $\mathbb{R}^2$ (planar systems)

The most important class of linear systems where the equation can be solved exactly is a **linear constant coefficient system** (LCC system)

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

where the matrix A is constant. To get the idea, we leave the general case in $\mathbb{R}^n$ to later and first consider **planar systems**,

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where } A \in \mathbb{R}^{2\times 2}. \tag{2.1}$$

It suffices to find two linearly independent solutions $\mathbf{x}_1, \mathbf{x}_2$. Once we do, solving an IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is simple since we need only find constants c_1, c_2 such that

$$c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) = \mathbf{x}_0$$

which is just the linear system

$$\begin{bmatrix}\mathbf{x}_1(t_0) & \mathbf{x}_2(t_0)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \mathbf{x}_0.$$

2.1. **The basics:** Obtaining the basis solutions is really just a linear algebra problem. Define

$$p(\lambda) = \det(A - \lambda I), \tag{2.2}$$

the **characteristic polynomial** whose roots are the eigenvalues of A. It is easy to check that if $\lambda, \mathbf{v}$ are an eigenvalue/vector pair for A then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution to (2.1). If the eigenvectors of A form a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ then

$$\{e^{\lambda_1 t}\mathbf{v}_1, \quad e^{\lambda_2 t}\mathbf{v}_2\}$$

is a basis for solutions to (2.1). In particular, this works when p has two distinct roots λ_1, λ_2.

Example (distinct real eigenvalues): Consider the system

$$\mathbf{x}'(t) = \begin{bmatrix}0 & 4\\ 1 & 0\end{bmatrix}\mathbf{x}.$$

The eigenvalues/vectors are

$$\lambda_1 = 2,\ \mathbf{v}_1 = (1,1)^T, \quad \lambda_2 = -2,\ \mathbf{v}_2 = (1,-1)^T$$

Thus the general solution is

$$\mathbf{x}(t) = c_1 e^{2t}(1,1)^T + c_2 e^{-2t}(1,-1)^T.$$

IVP: Now suppose we add an initial condition

$$\mathbf{x}(0) = (2,0)^T.$$

Plugging $t = 0$ into the general solution at and using the IC gives the system

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which is solved to obtain $c_1 = c_2 = 1$. Component-wise, the solution is

$$x_1(t) = e^{2t} + e^{-2t}, \quad x_2 = e^{2t} - e^{-2t}.$$

We see that $x_1, x_2 \to \infty$ as $t \to \infty$ and $x_1 \to \infty, x_2 \to -\infty$ as $t \to -\infty$ (exponentially).

2.2. **Complex roots:** First, a useful result:

Real/imaginary parts: If a complex function $\mathbf{z}(t)$ solves the linear system

$$\mathbf{z}'(t) = A(t)\mathbf{z}(t)$$

and $A(t)$ is real-valued, then the real and imaginary parts of $\mathbf{z}(t)$ are also solutions.

Short proof: write $\mathbf{z}(t)$ in terms of its real/imaginary parts,

$$\mathbf{z}(t) = \mathbf{z}_1(t) + i\mathbf{z}_2(t)$$

and substitute into the ODE to get

$$\mathbf{z}_1' + i\mathbf{z}_2' = A\mathbf{z}_1 + iA\mathbf{z}_2.$$

The real/imaginary parts on either side are equal, so $\mathbf{z}_1' = A\mathbf{z}_1$ and $\mathbf{z}_2' = A\mathbf{z}_2$ for all t.

Now suppose the matrix has complex eigenvalues (p has complex roots)

$$\lambda = r + \omega i, \quad \overline{\lambda} = r - \omega i$$

(why are they conjugates?) with eigenvectors

$$\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2, \quad \overline{\mathbf{v}} = \mathbf{v}_1 - i\mathbf{v}_2.$$

We want a basis for real solutions. The function

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a complex valued solution, but its real/imaginary parts will be two linearly independent basis solutions. We get

$$e^{\lambda t}\mathbf{v} = \left(e^{rt}e^{i\omega t}(\mathbf{v}_1 + i\mathbf{v}_2))\right) = e^{rt}(\cos\omega t\mathbf{v}_1 - \sin\omega t\mathbf{v}_2) + ie^{rt}(\cos\omega t\mathbf{v}_2 + \sin\omega t\mathbf{v}_1),$$

so a basis for real solutions is

$$e^{rt}(\cos\omega t\mathbf{v}_1 - \sin\omega t\mathbf{v}_2), \quad e^{rt}(\cos\omega t\mathbf{v}_2 + \sin\omega t\mathbf{v}_1).$$

Note that using the other eigenvalue $\overline{\lambda}$ yields the same results (check this!).

Remark: Of course if we wanted a basis for complex solutions, then $\{e^{\lambda t}, e^{\overline{\lambda}t}\}$ would be fine, with no further work.

Example (complex eigenvalues): The system

$$x' = y, \quad y' = -x$$

is of the form $\mathbf{x}' = A\mathbf{x}$ where $\mathbf{x} = (x, y)$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues/vectors are

$$\lambda = i \quad \mathbf{v} = (1, i)^T$$

and $-i$ and $(1, -i)^T$. Write the first eigenvectors in terms of real/imaginary parts:

$$\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2, \quad \mathbf{v}_1 = (1, 0),\ \mathbf{v}_2 = (0, 1).$$

One complex solution is

$$\mathbf{x}(t) = e^{it}\mathbf{v} = e^{it}(\mathbf{v}_1 + i\mathbf{v}_2).$$

Taking real/imaginary parts, we get two solutions:

$$\mathbf{x}_1 = \cos t\mathbf{e}_1 - \sin t\mathbf{e}_2 = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix},$$

$$\mathbf{x}_2 = \sin t\mathbf{e}_1 + \cos t\mathbf{e}_2 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

It is easy to verify they are linearly independent (plug in $t = 0$). The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

Now suppose we want to solve the IVP

$$x(0) = x_0, \quad y(0) = y_0$$

Since $\mathbf{x}(0) = (c_1, c_2)^T$ the linear system is easy to solve; $c_1 = x_0$ and $c_2 = y_0$. The solution moves clockwise on a circle of radius $\sqrt{x_0^2 + y_0^2}$ starting at (x_0, y_0).

2.3. **Repeated roots.** Suppose now that there is a single repeated eigenvalue λ. There are two cases:

i) There is a basis of eigenvectors $\mathbf{v}_1, \mathbf{v}_2$
ii) The eigenspace is deficient (only one eigenvectors $\mathbf{v}_1$)

Case (i) is easy: $e^{\lambda t}\mathbf{v}_1$ and $e^{\lambda t}\mathbf{v}_2$ are two linearly independent solutions.

For Case (ii), the procedure is given here and justified later. We have one solution,

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1.$$

To get the second, find a vector $\mathbf{v}_2$ (a **generalized eigenvector**) such that

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

This vector satisfies

$$(A - \lambda I)^2\mathbf{v}_2 = 0, \quad (A - \lambda I)\mathbf{v}_2 \neq 0.$$

The second solution is then

$$\mathbf{x}_2(t) = e^{\lambda t}(\mathbf{v}_2 + t(A - \lambda I)\mathbf{v}_2) = e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

which can be checked explicitly (we'll see why this is the case later).

Example (repeated, but easy): The system

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

has one eigenvalue $\lambda = 1$ and eigenvectors $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$. The solution is

$$\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = e^{\lambda t}(c_1, c_2)^T$$

which makes sense, of course since this is just the trivial system $x' = x, y' = y$.

Example (repeated eigenvalue): We solve

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(1) = (1, e^2)^T \text{ where } \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

The characteristic polynomial is $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$ so there is one eigenvalue $\lambda = 2$. The only eigenvector is $\mathbf{v}_1 = (1, 2)^T$. Solve

$$(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$$

to get

$$\mathbf{v}_2 = (-1, -1)^T.$$

A solution basis is then

$$\{e^{2t}\mathbf{v}_1,\ e^{2t}(\mathbf{v}_2 + t\mathbf{v}_1)\}.$$

To solve the IVP we need c_1, c_2 such that

$$c_1e^2(1, 2) + c_2e^2(0, 1) = (1, e^2).$$

solving this linear system, we get $c_1 = 1$, $c_2 = -1$ so the solution is

$$\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{2t} \begin{bmatrix} -1 + t \\ -1 + 2t. \end{bmatrix}$$

3. Second order ODEs

Second-order ODEs (and higher order as well) can be converted to systems; the theory developed in the previous section can be used to solve them as well. Consider the second order ODE

$$y'' + p(t)y' + q(t)y = f(t) \tag{3.1}$$

and associated IVP

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = a,\, y'(t_0) = b. \tag{3.2}$$

This is a system for (y, y'). To convert to a system, define

$$x_1 = y, \quad x_2 = y'.$$

Then (3.1) becomes the two-dimensional system

$$x_1' = x_2, \quad x_2' = f - qx_1 - px_2.$$

Letting $\mathbf{x} = (x_1, x_2)$, this is in the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$$

where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

The existence theorem, dimension of solution space and variation of parameters all apply; to get back to y, simply take the first component of $\mathbf{x}(t)$. Thus the solutions to (3.1) form a vector space with a basis of two solutions y_1, y_2. These solutions are linearly independent in the sense of the associated system:

Definition (linear independence of solutions) Two solutions y_1, y_2 to the ODE (3.1) are said to be 'linearly independent' if

$$\begin{bmatrix} y_1 \\ y_1' \end{bmatrix}, \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} \text{ are linearly independent vectors for all } t. \tag{3.3}$$

This is equivalent to 'at a point t_0' by the lemma from earlier.

The Wronskian is

$$W = y_1 y_2' - y_1' y_2$$

so to check that y_1, y_2 are linearly independent (i.e. a basis), we simply need to find a point t_0 such that $W(t_0) \neq 0$. That is, if y_1, y_2 are solutions then

$$\{y_1, y_2\} \text{ are a basis } \iff W(t_0) \neq 0 \text{ for some } t_0.$$

Of course, it is then true that $W(t) \neq 0$ for all t.

In general, we define linear independence of **functions** as follows:

Definition (linear independence of functions): Two functions y_1, y_2 are called linearly independent if

$$c_1 y_1 + c_2 y_2 = 0 \implies c_1 = c_2 = 0.$$

It turns out that two solutions are linearly independent in the sense of (3.3) if and only if they are linearly independent functions. This requires proof, however (which we defer to the more general case later). It is **not** true that the two definitions are equivalent if y_1, y_2 are not solutions to the ODE (see homework for examples).

3.1. **Constant coefficient case.** Here we solve

$$ay'' + by' + cy = 0. \tag{3.4}$$

As a system, the matrix is $A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$ which has characteristic polynomial

$$p(\lambda) = a\lambda^2 + b\lambda + c.$$

It is important to note that this equation inherits all the theory for the associated system. However, because of the nice structure, we can say more (and solve the equation more easily).

Let L be the linear operator for this ODE. For any λ,

$$L[e^{\lambda t}] = a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = p(\lambda)e^{\lambda t}.$$

We call $e^{\lambda t}$ an **eigenfunction** of L with eigenvalue $p(\lambda)$. In particular,

$$e^{\lambda t} \text{ is a solution iff } p(\lambda) = 0.$$

The basis solutions depend on the roots of $p(\lambda)$. Note that in all cases, we must check the solutions form a basis via the Wronskian.

3.2. **Solution procedure.** The process of solving second-order ODEs is the same as for planar systems, but a bit easier in the repeated roots case (plus, no eigenvectors to worry about). The case work is the same, since it deals with the same eigenvalues.

Real roots: If p has distinct real roots then

$$\{e^{\lambda_1 t}, \quad e^{\lambda_2 t}\}$$

form a basis.

Complex roots: If the eigenvalues are complex, then $\lambda = r \pm \omega i$. Taking real/imaginary parts of $e^{\lambda t}$ gives the basis solutions

$$e^{rt}\sin\omega t, \quad e^{rt}\cos\omega t$$

which is much easier than for systems because we are working with scalar functions.

Repeated roots: If there is a single eigenvalue λ, then the basis is

$$e^{\lambda t}, \quad te^{\lambda t}.$$

There are a number of ways to derive this (see homework). One way is to use the solution for the system,

$$\mathbf{x} = e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

and then take the first component.

Some examples:
Oscillation: The equation
$$y'' + \omega^2 y = 0, \qquad \omega \in \mathbb{R}$$
has characteristic polynomial
$$\lambda^2 + \omega^2$$
with roots $\pm\omega i$. A complex basis is then $e^{\pm i\omega t}$. Taking real and imaginary parts, we get the rela basis $\sin\omega t, \cos\omega t$ so
$$y = c_1 \sin\omega t + c_2 \cos\omega t.$$
The solution oscillates with frequency ω. This equation describes a simple harmonic oscillator - the most fundamental of systems that oscillate.

Repeated root: Consider the IVP
$$4y'' - 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 3/2.$$
The characteristic polynomial is $p(\lambda) = (2\lambda - 1)^2$ so two linearly independent solutions are $e^{t/2}$ and $te^{t/2}$. The general solution is
$$y = (c_1 t + c_2)e^{t/2}.$$
Plugging in $y(0) = 1$ we get $c_2 = 1$; then $y'(0) = 3/2$ gives $c_1 = 1$.

4. Loose ends: some theory and definitions

4.1. **Useful basis-related definitions.** When manipulating general solutions, there are a few quantities that are useful to have.

Fundamental matrix: The basis is also called a **fundamental set** for the ODE. It will also be useful to define the **fundamental matrix**
$$\Phi(t) = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$
which is just the matrix whose columns are the basis solutions. Note that in our construction, the initial conditions for the basis solutions are the columns of $\Phi(t_0)$. Typically, they are chosen so $\Phi(t_0) = I_n$, in which case
$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \implies \mathbf{x}(t) = \Phi(t)\mathbf{x}_0.$$

Wronskian: The **Wronskian** $W(t)$ of the ODE is defined as
$$W(t) = \det(\Phi(t)) = \det(\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}).$$
From linear algebra we know that
$$\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots \mathbf{x}_n(t) \text{ are linearly independent at } t \iff W(t) \neq 0.$$
So in terms of W, the linear independence lemma states that
$$W(t_0) \neq 0 \implies W(t) \neq 0 \text{ for all t }.$$
The Wronskian gives a succinct 'test' for linear independence (compute $W(t)$ at one point).