MATH 356 LECTURE NOTES
FIRST ORDER ODES:
FIRST ORDER ODES AND SOME FUNDAMENTALS

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Topics covered

- Basic definitions and concepts
  - Ordinary differential equations; initial value problems
  - What is a solution?
  - Separable equations
- Existence and uniqueness for initial value problems
  - finding the interval of existence; properties
  - The main existence theorem, extension theorem
  - Ways that a solution can fail to exist, non-uniqueness
- Exact equations
  - Exact differentials and potentials
  - Solving exact equations
  - Connection to conservative vector fields

1. What is a differential equation?

A differential equation is an equation that relates a **function** and its **derivatives**. Such equations are ubiquitous in the sciences, where physical systems depend on the rates of changes of quantities (electromagnetic waves, population growth, motion of planets, chemical reactions and so on).

Definition: An **ordinary differential equation** (ODE) relates a function $y(t)$ of one variable to its derivatives. We consider equations that can be written in the form

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \tag{1.1}$$

for some function F. The highest derivative, n, is the **order** of the ODE.

It is important to note that this is a relation between *functions* defined on some interval; solving it will require getting the whole solution all at once and we cannot simply solve for $y(t)$ at each t. To illustrate the point, consider

$$y(t) = t + y(t)^2 \tag{1.2}$$

$$z'(t) = t + z(t)^2. \tag{1.3}$$

The first is not an ODE. We can solve it by solving for $y(t)$ at each t via the quadratice formula:

$$y(t) = \frac{1 \pm \sqrt{1-4t}}{2}.$$

For the second, it is not so easy. The only way to simply 'undo' the derivative is to integrate, which gives

$$z(t) = \frac{1}{2}t^2 + \int z(t)^2\,dt$$

but $z(t)$ is an unknown function, so integrating in this way does not help. Solving this equation requires some new tricks.

In contrast to an ODE, a **partial differential equation** (PDE) is an equation for a function of more than one variable involving its *partial* derivatives. A few examples:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \text{ for } u(x,t) \quad \text{(Heat equation)}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ for } \phi(x,y). \quad \text{(Laplace's equation)}$$

The coupling between derivatives in several directions makes PDEs even more challenging to solve than ODEs. We will build up theory for ODEs first, and study PDEs later.

2. First order ODEs

To start building up the theory, we focus on the first order ODE

$$y' = f(t,y) \tag{2.1}$$

and the **initial value problem** (IVP)

$$y' = f(t,y), \qquad y(t_0) = y_0. \tag{2.2}$$

To start, we should clearly state what it means to be a solution:

What is a solution? A **solution** to the IVP (2.2) is a function $y(t)$ such that

i) $y(t)$ is defined in some interval (a,b) containing t_0 and $y(t_0) = y_0$

iii) $y(t)$ satisfies the ODE (2.1) in (a,b)

The 'solution' to the IVP comes with a domain where it is defined. The largest interval (a,b) where $y(t)$ is defined is called the **interval of existence**.

The **general solution** to the ODE (or 'solution' for short) is the most general expression for $y(t)$ that satisfies the ODE, including arbitrary constants.

Calculus analogy: The simplest first order IVP is one where the RHS depends on just t:

$$y' = f(t), \quad y(t_0) = y_0.$$

The general solution to the ODE can be written as

$$y(t) = C + \int_{t_0}^{t} f(s)\,ds.$$

for any t_0. An initial condition $y(t_0) = y_0$ would determine the constant of integration. The solution to the initial value problem is

$$y(t) = y_0 + \int_{t_0}^{t} f(s)\,ds.$$

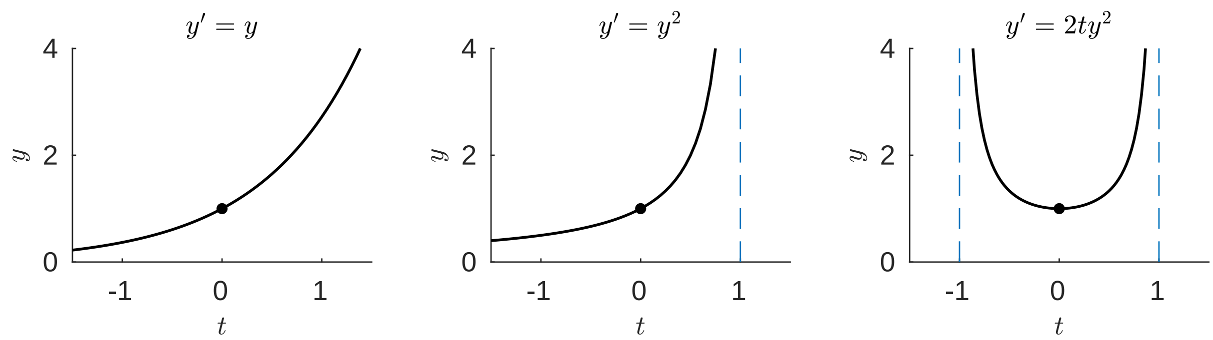


FIGURE 1. Solutions to the three example IVPs (i), (ii) and (iii). Dashed lines are asymptotes for the solutions.

When the right hand side depends on y, we cannot simply integrate.

You can think of the solution to an initial value problem as defined 'up to' b and 'down to' a, starting at t_0. For example, three IVPs and the general solution to the ODE and the solution to the IVP are listed below.

$$\begin{aligned} \text{i)} \quad & y' = y, \implies y(t) = Ce^t \\ & y(0) = 1 \implies y = e^t \\ \text{ii)} \quad & y' = y^2, \implies y(t) = \frac{1}{C-t} \\ & y(0) = 1 \implies y = \frac{1}{1-t} \\ \text{iii)} \quad & y' = 2ty^2, \implies y(t) = \frac{1}{C-t^2} \\ & y(0) = 1 \implies y = \frac{1}{1-t^2}. \end{aligned}$$

The solution for (i) is defined on all of $\mathbb{R}$. The solution for (ii), however, is defined only in $(-\infty, 1)$ because it diverges as t increases to 1. The solution for (iii) is defined only in the bounded interval $(-1, 1)$. Note that we cannot know this at a glance; the ODEs look similar and the asymptotes don't appear in the ODEs themselves.

2.1. **Geometric interpretation.** Solutions can be visualized as curves in the (t, y) plane using a **direction field**, which sometimes gives useful intuition. The equation

$$y'(t) = f(t, y)$$

tells us the rate of change of y for a solution curve $(t, y(t))$ at any point (t, y). The solution curve is tangent to the vector field

$$\mathbf{v}(t, y) = (1, f(t, y)).$$

We draw this vector field (the 'direction field') as arrows (ignoring the magnitude). All solutions must 'follow' this direction field. The advantage is that it is straightforward to draw; the downside is that it provides limited information.[1]

As an example, consider the **logistic equation**

$$y'(t) = y(1-y).$$

The direction field is shown with solutions curves from three starting points at $t = 0$ (for $y(0) = -0.5, 0.5$ and 1.5). From the plot, we can deduce what solutions must do as t increases to ∞ or decreases to $-\infty$. By following the direction field, the plot suggests

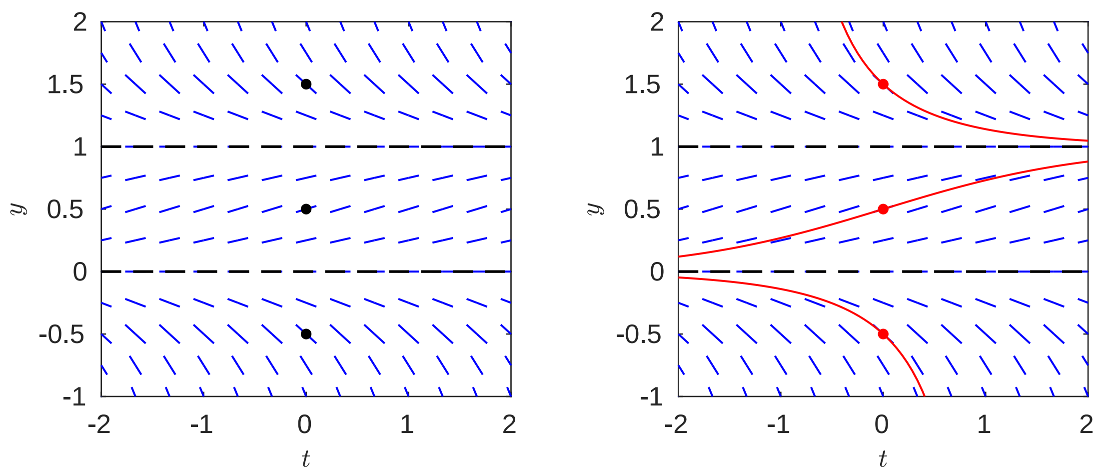
$$y_0 > 0 \implies \lim_{t\to\infty} y(t) = 1$$
$$y_0 < 0 \implies \lim_{t\to-\infty} y(t) = -1.$$

In the opposite directions, the solutions diverge to an asymptote, but this is not clear from the direction field - we need to solve the equation to get this information.

Equilibrium points: There are also two solutions that are constant:

$$y(t) \equiv 0 \text{ or } 1$$

which are **equilibrium points** (since the solution doesn't move). On the direction field, these are horizontal lines (where $y' = 0$ always. We'll study these special solutions in more depth later.



As a second example, consider

$$y' = -ty + 1, \quad y(0) = y_0.$$

The direction field is shown below. Observe that there are no equilibrium points. However, all solutions want to approach the line

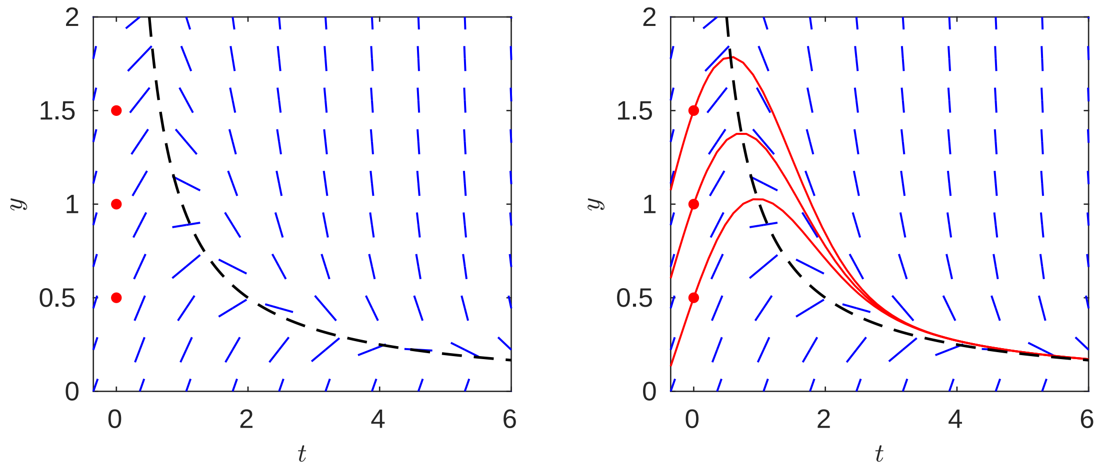
$$y_\ell(t) = 1/t$$

as $t \to \infty$. The direction field suggests this is true since

$$y' > 0 \text{ if } y > y_\ell, \quad y' < 0 \text{ if } y < y_\ell.$$

[1]Drawing the field is tedious by hand; we'll address some more efficient and powerful tools in Part II of the course. It is easy to plot a direction field in Matlab - an example can be found on the course website

Intuitively, the ODE wants to push solutions towards the line $y = 1/t$. Note that y_ℓ is not a solution to the ODE, unlike the equilibrium points of the first example.



3. Separable equations

Even first-order ODEs are complicated enough that exact solutions are not easy to obtain in general .One type that can be solved exactly is a **separable equation**, which is a first order ODE of the form

$$f(y)\frac{dy}{dx} = g(x) \tag{3.1}$$

for functions f, g. This can be integrated directly, if you recall the chain rule

$$\frac{d}{dx}\left(F(y(x))\right) = F'(y)\frac{dy}{dx}.$$

Let F, G be anti-derivatives of f, g (i.e. $F' = f$ and $G' = g$). Then (3.1) is

$$(F(y))' = -G'(x).$$

Now integrate, to find that solutions $y(x)$ satisfy

$$F(y(x)) - G(x) = C.$$

From here, we can leave the solution **implicit** or solve for y.

Example (solving separable equations): We find the general solution to the ODE

$$y' = xy^2 + x.$$

Rearrange:

$$\frac{y'}{1+y^2} = x.$$

Now integrate in x, using that $\int 1/(1+y^2)\,dy = \tan^{-1}(y)$,

$$\tan^{-1}(y(x)) = \frac{1}{2}x^2 + C.$$

Finally, we can solve for y:

$$y(x) = \tan(x^2/2 + C).$$

For the initial value problem

$$y' = xy^2 + x, \qquad y(1) = 1,$$

Plug in $x = 1$ and $y = 1$ to solve for C:

$$\tan^{-1}(1) = 1/2 + C \implies C = \pi/4 - 1/2.$$

Note that other values of C could be chosen, but this does not change the end result. The solution is

$$y(x) = \tan(x^2/2 + \pi/4 - 1/2).$$

Notational shortcut: The traditional shortcut is to write

$$\frac{1}{1+y^2}\, dy = x\, dx$$

and then 'integrate' both sides:

$$\int \frac{1}{1+y^2}\, dy = \int x\, dx + C.$$

This is, of course, equivalent to the previous solution, where we used

$$\int \frac{1}{1+y^2} \frac{dy}{dx}\, dx = \int x\, dx.$$

Multiplying both sides by dx is convenient but introduces some abuse of notation.

3.1. **intervals of existence: from the solution.** We have seen that solutions can have limited intervals of existence. Now that we have a method to obtain solutions, let's look at an example where we must solve the ODE to know where solutions are defined. Consider the IVPs

$$\begin{aligned} y' &= -(1-2x)y\ , & y(0) &= -1/2 & &(A)\\ y' &= \quad (1-2x)y^2, & y(0) &= -1/2 & &(B)\\ y' &= \quad (1-2x)y^2, & y(0) &= 4 & &(C). \end{aligned}$$

On what interval are these solutions defined? Unlike Example 1, the ODE function that gives y' is well-defined eveywhere, so there is no problematic value of y. We need to solve the equations to determine where solutions might fail to exist. For (A),

$$\frac{y'}{y} = -(1-2x), \quad \ln|y| = -x + x^2 + C.$$

Solving for $|y|$, we get

$$y = \pm C e^{x^2 - x}.$$

Since $y(0) = -1/2$, we should take the $-$ sign and $C = 1/2$, so the solution is

$$y = -\frac{1}{2} e^{x^2 - x}.$$

Even though $|y|$ grows very fast, this is defined for all $x \in \mathbb{R}$.

For (B), we have

$$\frac{y'}{y^2} = 1 - 2x \implies -\frac{1}{y} = x - x^2 + C \implies y(x) = \frac{1}{(x+1)(x-2)}.$$

The interval of existence is thus $(-1, 2)$.

Be careful: the general solution is the same for any initial condition, but the interval of existence depends on t_0 and y_0. For (C),

$$\frac{y'}{y^2} = 1 - 2x, \quad y(0) = 4 \implies y(x) = \frac{1}{x^2 - x + 1/4} = \frac{1}{(x - 1/2)^2}.$$

This solution has one asymptote at $x = 1/2$, so the solution is defined in $(-\infty, 1/2)$.

Remark: In (A), y' scales with y, which yields something like exponential growth. In (B), y' scales with y^2, causing it to grow *much* faster - so fast that it 'blows up' in a finite interval.

3.2. **interval of existence: using geometry.** The fact the solutions lie on level sets of

$$\phi(x, y) = F(y) - G(x)$$

can give useful geometric information about solutions. Consider the IVP

$$y' = -\frac{x}{y}, \qquad y(0) = R$$

for a constant R. This is separable:

$$yy' = -x \implies x^2 + y^2 = C.$$

Solutions therefore follow arcs of circles (level sets of x^2+y^2). Applying the initial condition,

$$y = \begin{cases} \sqrt{R^2 - x^2} & \text{if } R > 0 \\ -\sqrt{R^2 - x^2} & \text{if } R < 0 \end{cases}.$$

The interval of existence for $y(x)$ is $(-R, R)$ (note that we do not include $x = \pm R$ because the ODE itself is not well-defined there, even if $y(x)$ is).

We can read from the ODE that solutions may fail to exist, since $|y'| \to \infty$ as $y \to 0$ (assuming $x \neq 0$). The ODE determines a barrier solutions may not cross, and indeed this is clear from a contour plot of $x^2 + y^2$ (draw it!) - if solutions were to continue along the circle, then $y(x)$ would cease to be a function.

The x-interval on which $y(x)$ is defined depends on the initial condition; to find it, set $y = 0$ in the solution. This gives $x = \pm R$, as expected.

Highlight: This example shows that sometimes, we can deduce the value of y where solutions fail to exist from the ODE (without solving it). To get the interval (the x-values), we still need to do more work.
In both examples, the endpoints of the interval of existence were where $|y'| \to \infty$. However, $y' = \infty$ could only be deduced from the ODE in the second case.

4. Existence and uniqueness: motivation

Consider the first order initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{4.1}$$

Some fundamental questions to ask of this problem are:

- Under what conditions do solutions exist?
- Is the solution unique?
- How large is the interval of existence (where do solutions exist)?

Up until this point, we have only answered these questions by computing solutions explicitly. In this section, we provide some more general answers and gain some intuition for the issue of existence.

4.1. Examples.

A few examples will illustrate the potential 'problems' (where solutions fail to exist or fail to be unique). For contrast, recall that

$$y' = y, \quad y(0) = y_0$$

has a unique solution $y(t) = y_0 e^t$ for any y_0 whose interval of existence is $\mathbb{R}$.

Ex. 1 (small interval of existence): Consider the IVP

$$y' = xy^2, \quad y(0) = y_0 > 0.$$

By separating variables, it is easy to show that the solution is

$$y = \frac{1}{1/y_0 - x^2}.$$

Note that the solution process guarantees that the solution is unique since the steps can be reversed. The solution exists in the interval $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$, which can be arbitrarily small (depending on y_0).

Ex. 2 (No solution or infinitely many?): Consider the IVP

$$ty' = y \text{ for } t > 0, \quad y(0) = y_0.$$

To be precise, note that a solution is a function $y(t)$ that satisfies the ODE for all $t > 0$ and has $y(0) = y_0$.

The general solution to the ODE is

$$y = Ct.$$

If $y_0 \neq 0$ then there are no solutions. If $y_0 = 0$ then $y = Ct$ is a solution for any constant C - an infinite set of solutions. Note that the interval of existence is $[0, \infty)$; we need to modify the definition slightly because of the $t > 0$ restriction.

In contrast, the IVP

$$2t^{1/2} y' = 1 \text{ for } t > 0, \quad y(0) = y_0$$

is trivial to solve and has the unique solution

$$y = y_0 + t^{1/2}.$$

Ex. 3 (Infinitely many solutions): A more dramatic example of an infinite set of solutions is the IVP

$$y' = 2y^{1/2}, \quad y(0) = 0.$$

One solution is $y = 0$, and by separating variables we find that

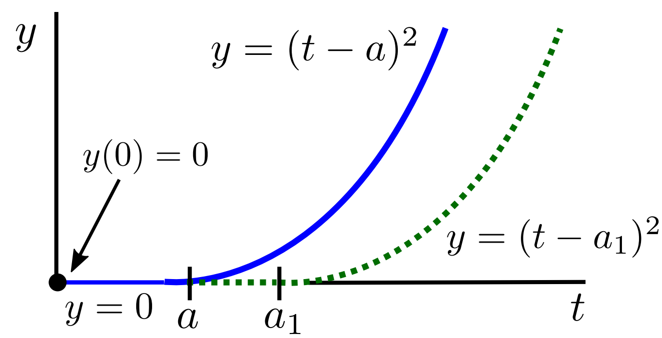
$$y = (t-a)^2$$

solves the ODE for any constant a. Now for any $a > 0$ we can glue together two solutions to the ODE to create a new solution

$$y = \begin{cases} 0 & t < a \\ (t-a)^2 & t \geq a. \end{cases}$$

It is easy to check that y and y' are continuous (despite the piecewise definition). We therefore have an infinite family of solutions for **any** $a \geq 0$ to the *initial value problem* that start equal to zero, then switch to become non-zero at any point $a \geq 0$.

This means that to have a unique solution, we would need an additional constraint to determine when the solutions leaves $y = 0$. For a physical scenario in which this issue arises naturally, see the homework.



Above: two possible piecewise solutions that become non-zero at a and a_1.

5. Existence and uniqueness: theory

What conditions on f give us some control over possible solutions to the ODE? The examples suggest that continuity and the dependence of f on y is also important. The key definition turns out to be the following:

Definition (Lipschitz): Let (a,b) be an interval (possibly infinite). A function $f:(a,b)\to\mathbb{R}$ is called **Lipschitz** if there is a constant $L>0$ (the 'Lipschitz constant') such that

$$|f(x)-f(y)|\leq L|x-y| \quad \text{for all } x,y\in(a,b).$$

Sufficient condition: If f is differentiable and

$$|f'(x)|\leq L \text{ for all } x\in(a,b)$$

then f is Lipschitz in (a,b) with constant L (to prove this, use the mean value theorem).

This is a stronger condition than continuity but weaker than 'bounded derivative', and it allows us to bound a change in a function by a change in its arguments *uniformly*.

We need a version of this condition for functions $f(t,y)$. As it will be used throughout, we will refer to it hereafter as 'condition L_y'.

Definition (the key property for this section): Let $R=(a,b)\times(c,d)$ be a rectangle in the (t,y) plane. Call 'condition L_y' the Lipschitz condition

$$|f(t,x)-f(t,y)|\leq L|x-y| \text{ for all } t\in(a,b) \text{ and } x,y\in(c,d). \tag{L_y}$$

That is, for each fixed t, $f(t,y)$ as a function of y is Lipschitz, all with the same constant.

Sufficient condition: If $\frac{\partial f}{\partial y}$ exists in the domain and

$$\left|\frac{\partial f}{\partial y}\right|\leq L \text{ for all } (t,y)\in R \tag{L_y'}$$

then the condition (L_y) holds. This is much easier to check in practice.

The Lipschitz condition (L_y) gives us a way of measuring how sensitive an IVP can be to changes in the initial conditions, which in turn will give us a nice condition for uniqueness. The technical details and proof will be omitted.

Lemma (dependence on initial conditions): Suppose $x(t)$ is a solution to

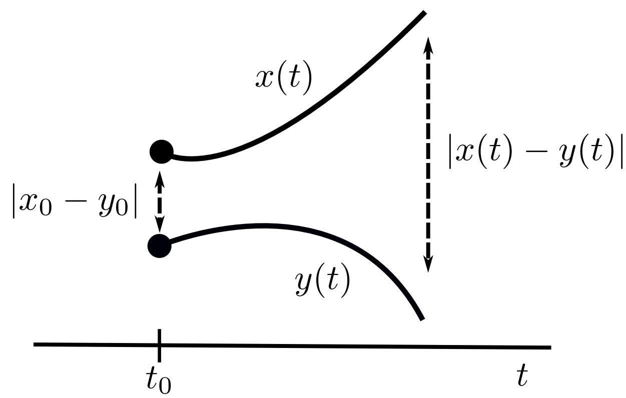
$$x'=f(t,x), \quad x(t_0)=x_0$$

and $y(t)$ is a solution to

$$y'=f(t,y) \quad y(t_0)=y_0.$$

If f satisfies (L_y) in a domain R containing (t_0,x_0) and (t_0,y_0) then

$$|x(t)-y(t)|\leq e^{L|t-t_0|}|x_0-y_0| \text{ for all } t \text{ s.t. the solutions stay in } R.$$



That is, two solutions to the same ODE can move apart at worst exponentially with rate L. Thus (when f is differentiable), the size of $\partial f/\partial y$ controls this sensitivity to changes in initial conditions.

As an immediate corollary, we obtain a uniqueness result:

Theorem (uniqueness): Consider the IVP

$$y' = f(t,y), \quad y(t_0) = y_0$$

where satisfies (L_y) in some region containing (t_0, y_0). Then if the solution to the IVP exists, it is unique where (L_y) holds (that is, unique at least close to t_0).

Proof. Suppose $y_1(t)$ and $y_2(t)$ are both solutions to the IVP. Then both exist at least in some small interval contained in (a,b) where (L_y) holds. By the lemma,

$$|y_1(t) - y_2(t)| \leq e^{L|t-t_0|}|y_0 - y_0| = 0$$

since both initial conditions are equal. Thus $y_1(t) = y_2(t)$ for all t where the Lemma applies. □

Idea of proof of lemma (extra): We have that

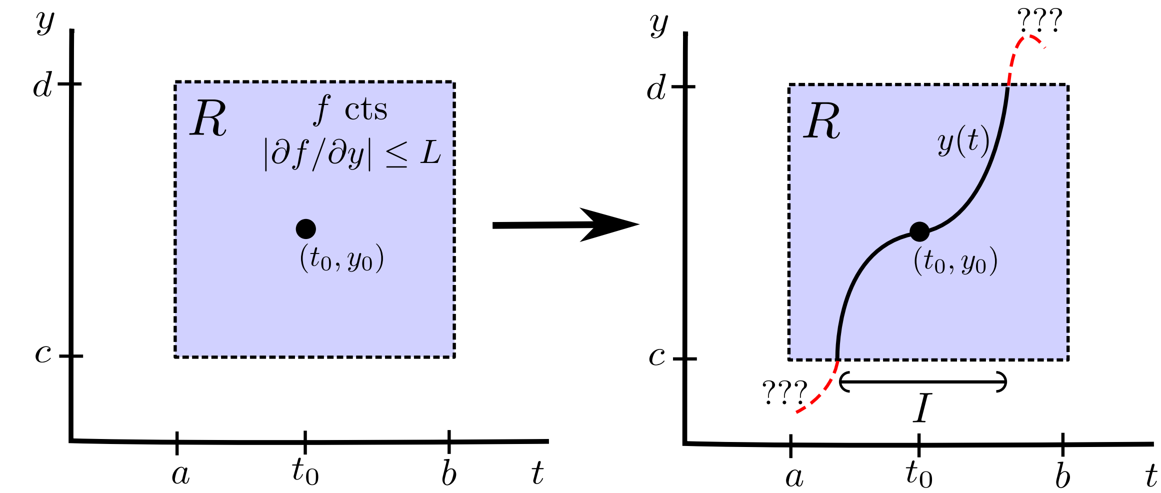
$$|(x-y)'| \leq |f(t,x) - f(t,y)| \leq L|x-y|.$$

Letting $z(t) = x(t) - y(t)$, we see that

$$|z'(t)| \leq L|z(t)|. \tag{5.1}$$

This says that $z(t)$ grows at most as fast as Lz. We know that $z' = Lz$ has solution $z(t) = z(t_0)e^{L(t-t_0)}$, so it is plausible that (5.1) implies that the difference $z = x - y$ grows at most exponentially with rate L.

5.1. **Main theorem.** With the Lipschitz condition defined, we can now state the main theorem addressing both existence and uniqueness[2]. For simplicity, we state it using the sufficient condition for (L_y) instead (this can be replaced with (L_y) itself).



(Existence/uniqueness Theorem): Consider the IVP

$$y' = f(t,y), \quad y(t_0) = y_0. \tag{5.2}$$

Let $R = (a,b) \times (c,d)$ be an open rectangle containing (t_0, y_0).

I (Existence): If

$$f(t,y) \text{ is continuous in } R \tag{C}$$

Then there is an open interval $I \subset (a,b)$ (possibly smaller) containing t_0 such that a solution $y(t)$ exists for the IVP (5.2) that is defined on I.

II (Uniqueness): The solution is unique if in addition (L_y) holds in R.

A sufficient condition for (II) (stronger, but easier to check) is that

$$\frac{\partial f}{\partial y} \text{ is continuous in } R. \tag{C_y}$$

Key point: The existence theorem provides only a **local solution**, which means it guarantees a solution only in some interval around t_0 that could be arbitrarily small. However, there is one more useful result that gives a bit more information about the size of I:

Extension theorem: If (C) and (L_y) hold then $y(t)$ exists at least until it leaves R.

That is, we can extend the solution **until it hits the bounds of the rectangle.** Thus we can take the interval of existence I to be the whole interval where y stays in R, but we need to know more about y to determine I explicitly.

[2]The theorem stated here is a version of the **Picard-Lindel of** theorem, one of the fundamental theorems in the study of ODEs. It is the main theorem for existence/uniqueness, although there are a variety of other more specific results.

5.2. **Applying the theorem.** Often, the result is straightforward to use. To simplify things, let use the 'theorem' that[3]

$$f \text{ cts. at } (t_0, y_0) + \text{ and 'reasonable'} \implies f \text{ cts. in a rectangle } R \text{ around } (t_0, y_0). \qquad (*)$$

You are welcome to assume f is reasonable enough that (*) is true for this course. Thus to show that a unique 'local solution' exists IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

without any information about the interval of existence, we proceed as follows:

- Check that $f(t, y)$ and $\partial f/\partial y$ are continuous at (t_0, y_0) and note that it is clear that (C) and (C_y) hold in some rectangle R containing (t_0, y_0).
- Invoke the existence/uniqueness theorem (parts I and II).

Alternately, to find all the initial conditions where the theorem applies

- Identify all the points where $f(t, y)$ and $\partial f/\partial y$ have discontinuities.
- Conclude that if (t_0, y_0) is bounded away from these discontinuities, (C) and (C_y) hold in a rectangle R containing (t_0, y_0).
- Use the existence/uniqueness theorem.

To get information about where $y(t)$ exists, we would construct an actual R where f and $\partial f/\partial y$ are continuous.

Simple example: Consider the IVP

$$y' = \frac{t}{y+1}, \quad y(t_0) = y_0.$$

What are all values of t_0 and y_0 such that a unique solution is guaranteed to exist?

Solution: The function $f(t, y)$ is only discontinuous at $(t_0, -1)$ for any t_0, with the same for

$$\frac{\partial f}{\partial y} = -t/(y+1)^2.$$

If $y_0 \neq -1$ then there is clearly a rectangle R around (t_0, y_0) such that f and $\frac{\partial f}{\partial y}$ are continuous. Thus, by the theorem, a unique solution exists if $y_0 \neq -1$. *Note: the 'in some interval containing t_0' is built into our definition of 'solution', so you don't need to include this phrase except for emphasis.*

As an example that uses the extension theorem, consider the 'linear' IVP (to be studied in detail later)

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

where $p(t)$ and $g(t)$ are continuous in an interval $[a, b]$ containing t_0. In this case the 'ODE function' is $f(t, y) = g(t) - p(t)y$ and

$$\left|\frac{\partial f}{\partial y}\right| = |p(t)|.$$

[3]Most of the exceptions to this rule are pathological functions that are continuous only at one point. But such functions are not often considered in ODEs, so it does not come up much.

Since $p(t)$ is continuous on a closed interval, it is bounded by some M in $[a, b]$, so

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq M \text{ in } [a, b] \times \mathbb{R}.$$

Obviously f is continuous also since p and g are continuous. Thus (C) and (C_y) hold in $R = [a, b] \times \mathbb{R}$. But this 'rectangle' contains all $y \in \mathbb{R}$, so the solution can only leave through the sides $t = a$ or $t = b$. The extension theorem then tells us that the solution must exist in all of (a, b).

Revisiting existence/uniqueness examples.
Now let's see how the theorem applies (or fails to apply) to the examples before:

Example 1: $y' = xy^2$ with $y(0) = y_0 > 0$. The ODE function $f(x, y) = xy^2$ is clearly continuous, as is $\frac{\partial f}{\partial y} = 2xy$.

However, any rectangle R such that $|\frac{\partial f}{\partial y}| \leq L$ (condition (L_y)) must be bounded in y; it cannot contain all $y \in \mathbb{R}$. This means we cannot use the extension theorem to find the interval of existence (the solution can always leave by y growing too large). This makes sense, since the correct interval of existence shrinks to zero width as $y_0 \to \infty$.

Example 2: This is straightforward; $f(t, y)$ is not continuous at $t = 0$ so neither part of the theorem applies.

Example 2 (last part): For the IVP

$$y' = \frac{1}{2t^{1/2}} \quad t > 0, y(0) = y_0$$

the ODE function is not continuous so the theorem does not apply. However, a solution exists anyway. This shows that (C) is a sufficient but not necessary condition, i.e. if it fails then it does **not necessarily follow that no solution exists.**

Example 4: For $y' = 2y^{1/2}$, condition (C) holds but (L_y) does not. Thus part (I) (existence) applies but part (II) (uniqueness) does not, consistent with the example.

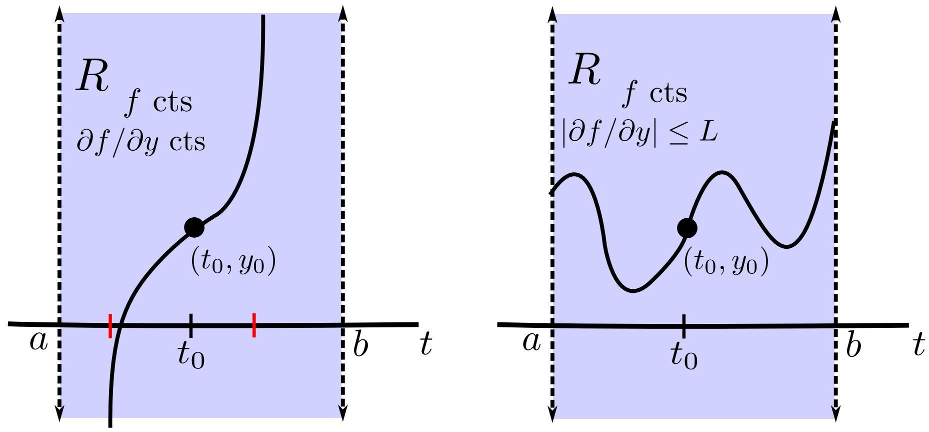
5.3. **Extension.** The theorem is a local result in that it does not give us information about the size of the solution interval. From the theorem, we know that if f is continuous then the interval extends until $y(t)$ leaves R. But we do not know $y(t)$, so this does not tell us the interval of existence.

However, suppose the 'rectangle' is instead

$$R = (a, b) \times \mathbb{R}$$

i.e. an infinite strip from $t = a$ to $t = b$. The example $y' = y^2$ shows that it is **not** enough to have that f and $\frac{\partial f}{\partial y}$ are continuous in R to guarantee the solutions extends to all of (a, b): it may diverge ($|y| \to \infty$) instead.

However, it turns out that a bound on $\frac{\partial f}{\partial y}$ is enough to ensure that solutions grow slow enough that they do not diverge. In this case, they must 'leave' R through the sides $t = a$ and $t = b$, ensuring existence in (a, b) (see figure below).



With some work, we can obtain the following result:

Theorem (Existence on an interval): If f is continuous for $t \in [a, b]$ and all $y \in \mathbb{R}$ and there is a constant L such that

$$\left|\frac{\partial f}{\partial y}\right| \leq L \text{ for all } t \in [a, b] \text{ and } y \in \mathbb{R}$$

then the solution exists in $[a, b]$.

By picking the right R, we can sometimes use this theorem to find the interval of existence. However, it requires a bound on $\frac{\partial f}{\partial y}$ for all y. Contrast

$$(i) \quad y' = \sin y$$

$$(ii) \quad y' = y^2.$$

For (i), we have $|\partial f / \partial y| = |\cos y| \leq 1$, so solutions to this ODE will exist for all t by the theorem (the condition holds in any strip $(a, b) \times \mathbb{R}$ with $L = 1$).

However, for (ii), $|\partial f / \partial y| = 2|y|$ is not bounded for all $y \in \mathbb{R}$, so we are stuck taking a rectangle R with finite y-bounds; (I) and (II) of the main theorem apply but we get no information about the interval of existence.

6. Consequences of uniqueness

6.1. **Solutions cannot cross.** Uniqueness constrains where solutions can go if the conditions of the theorem are satisfied. The most important consequence is the following:

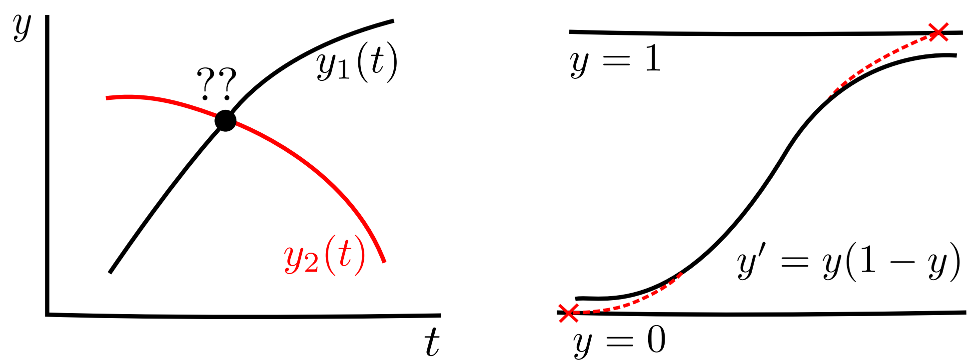
Non-intersecting solutions: If $y_1(t)$ and $y_2(t)$ are distinct solutions to the ODE

$$y' = f(t, y)$$

then the solution curves $(t, y_1(t))$ and $(t, y_2(t))$ **cannot intersect** at any point in the (t, y) plane where the uniqueness theorem applies.

The argument is simple (see left figure below): If the two curves intersected at a point (t_0, a), then they would both satisfy the same ODE with the same initial condition ($y_1(t_0) = y_2(t_0) = a$). Then y_1 and y_2 are distinct solutions to the same IVP - a contradiction.

Note that the fact that the existence theorem only gives a 'local' solution is not a problem: we need only guarantee uniqueness in an arbitrarily small neighborhood of any starting point for the argument to work.



We can sometimes use this to show bounds for solutions. As an example, observe that the ODE

$$y' = y(1-y)$$

has constant solutions $y \equiv 0$ and $y \equiv 1$. No other solution can cross these lines (see right figure above), which means that if $y(t)$ solves the ODE with initial condition

$$y(t_0) = y_0, \quad y_0 \text{ between 0 and 1}$$

then it must stay between 0 and 1 for all times where it exists:

$$0 < y_0 < 1 \implies 0 < y(t) < 1 \text{ for all } t.$$

When the uniqueness theorem fails: Typically the uniqueness theorem will apply; however, when it does not, we cannot conclude solutions do not intersect. For instance,

$$y' = \sqrt{1-y^2}$$

has constant solutions $y = -1$ and $y = 1$. But $\frac{\partial f}{\partial y}$ is infinite at $y = \pm 1$, so solutions do not have to be unique there. This is indeed the case; $y(t) = \sin(t + C)$ is a solution for any C, which touches both of the constant solutions.

6.2. **A physical example.** Infinite sets of solutions are not just mathematical oddities! Non-uniqueness often has something important to say about a model. Consider a bucket full of water with a hole at the bottom that starts with a height h_0 of water at $t = 0$. Using **Toricelli's law** from physics, the height of the water $h(t)$ at time t can be shown to satisfy an ODE of the form

$$\frac{dh}{dt} = -kh^{1/2}, \qquad h(0) = h_0.$$

where k is a constant depending on the geometry of the bucket. As in the earlier example, it is not hard to derive that

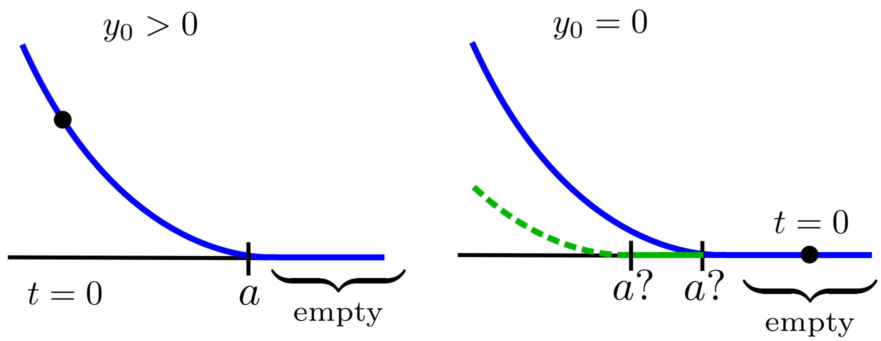
$$h(t) = \begin{cases} 0 & t > a \\ \frac{k^2}{4}(a-t)^2 & t < a \end{cases}.$$

is a solution for all 'emptying times' a. Here a represents the first time at which the height is zero, i.e. there is no water left in the bucket.

If the bucket is not empty at $t = 0$ ($h_0 > 0$), then the uniqueness theorem applies and we have a unique solution with

$$a = \sqrt{4h_0}/k.$$

However, if the bucket is empty at $t = 0$, then there is a solution $h_a(t)$ for all emptying times $a < 0$. That is, if the bucket is empty at $t = 0$, we cannot tell when it was full. This physical reality manifests in the model as the non-uniqueness of solutions.



7. Exact differentials

There is one more (important) type of first order ODE than can be solved explicitly. This has some connections to the theory for conservative vector fields you may recall from calculus.

7.1. Theory.

Consider a planar vector field ($\mathbb{R}^2 \to \mathbb{R}^2$)

$$\mathbf{v} = (M(x,y), N(x,y)).$$

Define the 'differential form'[4]

$$\omega = M\,dx + N\,dy. \tag{7.1}$$

Formally, ω is an object that can be integrated along a path Γ in the plane,

$$\Gamma = \{(x(t), y(t)) : t \in [t_0, t_1]\} \subset \mathbb{R}^2.$$

The integral of ω (by definition) is the line integral of the vector field along Γ:

$$\int_\Gamma \omega = \int_\Gamma (M\,dx + N\,dy) = \int_\Gamma \mathbf{v} \cdot d\mathbf{x} = \int_{t_0}^{t_1} \left(M\frac{dx}{dt} + N\frac{dy}{dt} \right) dt.$$

Exact differentials: A differential form (7.1) is called **exact** if there is a function $\phi(x,y)$ such that

$$d\phi = M\,dx + N\,dy.$$

This formal expression means that along any path $(x(t), y(t))$,

$$\frac{d\phi}{dt} = M\frac{dx}{dt} + N\frac{dy}{dt}.$$

To characterize exact differentials, recall some important results in vector calculus:

Definition: A vector field $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ is called **conservative** if

$$\mathbf{v} = \nabla\phi \text{ for a function } \phi : \mathbb{R}^n \to \mathbb{R}. \tag{7.2}$$

The function ϕ is sometimes called a 'potential' (often an actual potential in physics, e.g. ϕ = voltage and $\mathbf{v}$ = electric field). Conservative vector fields are gradients of a potential.

Conservative vector fields are also characterized by **path independence** (not needed here): the line integral of $\mathbf{v}$ over any curve Γ depends only the value of the endpoints: $\int_\Gamma \mathbf{v} \cdot d\mathbf{x} = \phi(\Gamma(t_1)) - \phi(\Gamma(t_0))$ where $\Gamma(t)$ runs from $t = t_0$ to $t = t_1$.

Theorem: Let $D \subset \mathbb{R}^3$ be a simply connected domain (a domain with no holes). A vector field $\mathbf{v} : D \to \mathbb{R}^3$ is **conservative** (i.e. (7.2) holds) if and only if

$$\nabla \times \mathbf{v} = 0 \text{ in } D.$$

[4] Note: the definition is much more general (see **Stokes' theorem**); here we are only considering a specific kind. In one dimension, $f(x)\,dx$ is also a differential form, which of course can be integrated: $\int f(x)\,dx$. Every $f(x)\,dx$ is exact since it equals $dF = F'(x)\,dx$ for the anti-derivative $F = \int f$.

That is, a vector field in a nice domain of $\mathbb{R}^3$ is the gradient of a function iff it has zero curl.

Corollary: A planar vector field $\mathbf{v} = (M(x,y), N(x,y))$ is conservative in D iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ in } D. \tag{7.3}$$

7.2. **Exact ODEs.** We determine when a differential form (7.1) is exact. By the chain rule,

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy$$

so it follows that $\omega = M\, dx + N\, dy$ is exact if and only if

$$\begin{aligned} (M,N) &= (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}) \text{ in } D \\ \iff \mathbf{v} &= \nabla\phi \text{ in } D \\ \iff \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \text{ in } D. \end{aligned}$$

That is, ω is exact in a region D if and only if (7.3) holds (the vector field is conservative).

Exact equations: An **exact equation** is an ODE

$$0 = d\phi.$$

This ODE describes **the level sets of** ϕ, i.e. the curves $\{\phi = \text{const.}\}$. From the theory above, we see that a general first order ODE

$$M(x,y)\, dx + N(x,y)\, dy = 0 \tag{7.4}$$

is exact if and only if (7.3) holds ($\partial M/\partial y = \partial N/\partial x$).

7.3. **Computing solutions.** Now let us suppose we have a general first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

or, more generally, a differential form

$$\omega = M\, dx + N\, dy.$$

If the exactness condition (7.3) holds, then we can easily compute ϕ by a simple integration.

Finding the potential: If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then find ϕ such that $\nabla\phi = (M,N)$ as follows:

1) Integrate $M = \frac{\partial \phi}{\partial x}$ to obtain ϕ up to an **arbitrary** function of y:

$$\phi = \int M\, dx + g(y). \tag{A}$$

2) Differentiate (A) with respect to y to get

$$N = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(\cdots) + g'(y) \implies g'(y) = \cdots \tag{B}$$

3) Solve this equation for $g(y)$. If the equation is exact, this step will work; (B) will only have the y variable (an **ODE** in y, no x).
4) The potential ϕ is given by (A), unique up to an arbitrary constant.

Practical note: One could also integrate in y first if this is easier.

Example: We find the general solution to

$$2xy + (2y + x^2)y' = 0$$

and the interval of existence. First check that this is exact:

$$\frac{\partial}{\partial y}(2xy) = 2x, \quad \frac{\partial}{\partial x}(2y + x^2) = 2x.$$

Step (1): integrate the first term ($M = 2xy$) in x:

$$\phi = \int 2xy\, dx = x^2 y + g(y).$$

Step (2-3): Differentiate in y and set equal to the coeffiicent of y':

$$2y + x^2 = \frac{\partial \phi}{\partial y} = x^2 + g'(y) \implies 2y = g'(y) \implies g = y^2 + C.$$

The potential is thus $\phi = x^2 y + y^2$. Solutions $y(x)$ to the ODE satisfy the implicit relation

$$\phi = x^2 y + y^2 = C. \tag{7.5}$$

Interval of existence: The solution fails to exist where $y' \to \infty$. From the ODE

$$y' = -2xy/(2y + x^2),$$

we see that solutions fail to exist where $x^2 = -2y$. Now substitute this into (7.5):

$$-x^4/2 + x^4/4 = C \implies x = \pm(-4C)^{1/4}$$

so the interval of existence is $\mathbb{R}$ if $C \geq 0$ and $(-(4C)^{1/4}, (-4C)^{1/4})$ if $C < 0$. Each value of C yields two solutions. If $C = 0$ the solutions are $y = 0$ and $y = x^2$, so the IVP with $y(0) = 0$ does not have a unique solution (why does this make sense?).

7.4. **Example (with level sets of ϕ):** We solve the IVP

$$2x - y + (2y - x)y' = 0, \qquad y(0) = \sqrt{3}/2$$

and determine the interval of existence. First, check that it is exact:

$$\frac{\partial}{\partial y}(2x - y) = -1, \quad \frac{\partial}{\partial x}(2y - x) = -1.$$

Step (1): Integrate $M = 2x - y$ in x:

$$\phi = x^2 - xy + g(y).$$

Step (2): Differentiate with respect to y and use that $\frac{\partial \phi}{\partial y} = 2y - x$:

$$2y - x = \frac{\partial \phi}{\partial y} = -x + g'(y) \implies 2y = g'(y) \implies g = y^2$$

Thus $h(x) = y^2$ so the general solution is

$$x^2 - xy + y^2 = C.$$

Solutions to the ODE are arcs of an ellipse (see Figure). Plug in $(0, \sqrt{3}/2)$ to get $C = 3/4$ for the IVP, so the solution to the IVP satisfies

$$x^2 - xy + y^2 = 3/4, \quad y(0) = \sqrt{3}/2. \tag{7.6}$$

The solution fails to exist where $|y'| \to \infty$; it is not the whole ellipse. The ODE says this occurs when $y = x/2$; plugging into (7.6) gives

$$3x^2/4 = 3/4 \implies \text{solution is defined for } |x| < 1.$$

The solution curve (red) and some contours of ϕ are shown below.