MATH 356 LECTURE NOTES
NONLINEAR SYSTEMS
PHASE PLANES: DEFINITIONS, FUNDAMENTALS

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Topics covered

- Nonlinear systems and the phase plane
  - Drawing the phase plane (nullclines, equilibria)
  - Periodic solutions; closed orbits
  - Invariant sets, stable and unstable manifolds
  - Example of drawing a phase plane (version 1)
- Equilibria and linearization
  - Stability definitions (asymptotic, Lyapunov, unstable)
  - Connection to linear case (spirals, nodes etc.)
  - When linearization works and when it doesn't (main theorem)
  - Drawing phase planes (version 2, with linearization); detailed example

1. Non-linear planar systems

With LCC systems detailed, we now move on to the general **non-linear** planar system[1]

$$x' = f(x,y), \quad y' = g(x,y) \tag{S}$$

and, with $\mathbf{x} = (x,y)$ and $\mathbf{F} = (f,g)$, the vector form

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}).$$

The path of a solution in the phase plane is called a **solution curve** or an **orbit**. Recall that because the system is autonomous, we can refer to 'the solution' through a point (x_0, y_0) in the phase plane (regardless of starting time t).

The existence/uniqueness theorem for first-order ODEs can be extended to systems:

Theorem (existence/uniqueness): Consider the initial value problem (in $\mathbb{R}^n$)

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Suppose, in a region R around $\mathbf{x}_0$,

i) $\mathbf{F}(\mathbf{x})$ is continuous in R

ii) The partial derivatives of $\mathbf{F}$, i.e. $\partial F_i/\partial x_j$, are continuous in R

Then a unique solution exists in a t-interval (a,b) containing t_0, extending until it leaves R.

Corollary: If (i) and (ii) hold everywhere, the solution exists until it diverges ($|\mathbf{x}| \to \infty$). Thus, orbits in the phase plane exist until they diverge to ∞.

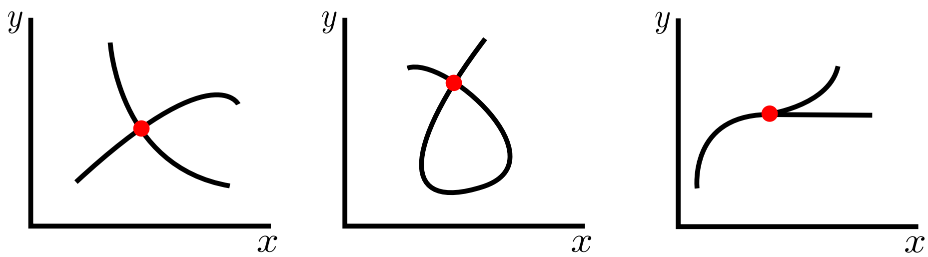
[1]The study here will be introductory - a mostly informal look at the rich subject of non-linear dynamics in the phase plane. For much more, an excellent reference is Strogatz' *Nonlinear Dynamics and Chaos*.

Non-intersection: A key consequence of uniqueness is that **distinct orbits cannot intersect.** This property goes a long way in deducing solution behavior in the phase plane (just as on the phase line).

Proof. Suppose two solution curves $\mathbf{x}_1$ and $\mathbf{x}_2$ intersect at a point $\mathbf{x}_0$. Both solve the IVP

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

so they must be the same solution by the uniqueness theorem. □



intersections not allowed!

Periodic solutions: A **periodic solution** (with period T) is a solution such that $\mathbf{x}(t) = \mathbf{x}(t+T)$ for all t. That is, it repeats after a 'period' T.

Closed orbits: A periodic solution in the phase plane forms a closed (a '**closed orbit**'). For example, the system

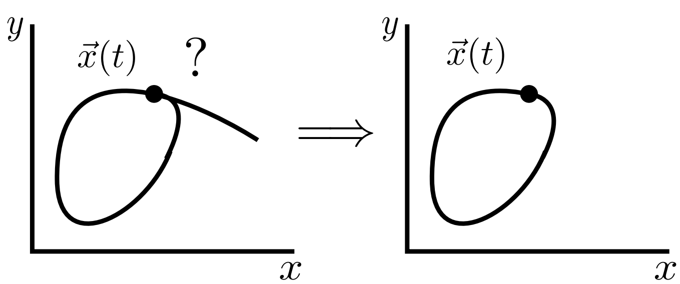
$$x' = y, \quad y' = -x$$

has a center at $(0,0)$; solutions are $x(t) = c_1 \cos t$, $y(t) = -c_1 \sin t$. The solutions form closed orbits that are circles with period 2π.

Self-intersection: The non-intersection property also implies that

if an orbit intersects itself, it must be a closed orbit

as shown in the diagram below. We can use this to deduce a a closed orbit exists; it suffices to show that $\mathbf{x}(t)$ intersects itself at some point.



self-intersection $\Longrightarrow$ closed orbit

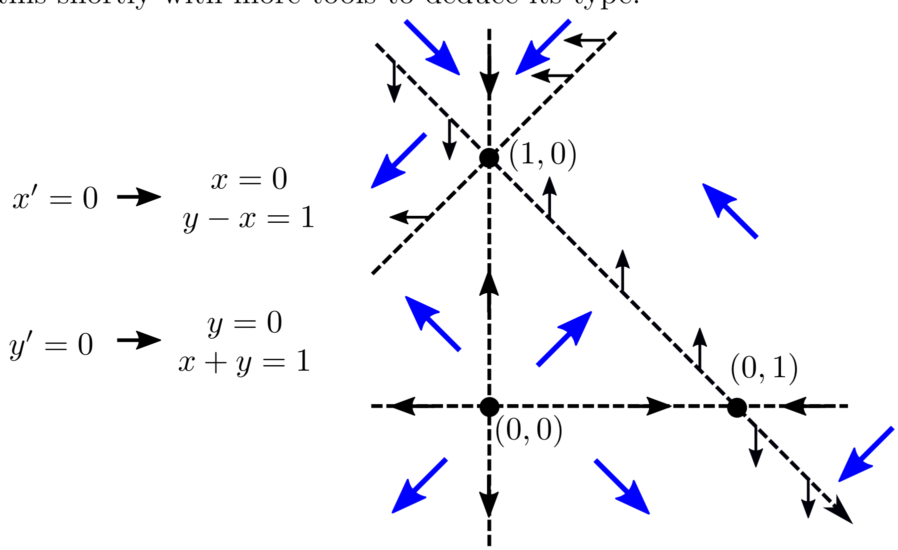
1.1. **Equilibria and nullclines.** A (non-degenerate) LCC planar system $\mathbf{x}' = A\mathbf{x}$ can have only one equilibrium at $(0,0)$. The system (1) can have many equilibria, and much more complicated structure. To reiterate some definitions:

- **Equilibria:** An **equilibrium point** (or **fixed point**) is a point $\mathbf{x}^*$ such that
$$F(\mathbf{x}^*) = 0. \tag{1}$$
The equilbria are exactly the constant solutions $\mathbf{x}(t) = \mathbf{x}^*$.

- **Nullclines:** For nonlinear systems, nullclines are not always straight lines. The x and y nullclines are the set of points in the phase plane where x' or y' are zero, respectively. That is, for (S) they are the sets
$$\{x' = 0\} = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\},$$
$$\{y' = 0\} = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}.$$
Solution curves cross the x**-nullcline vertically** and cross the y**-nullcline horizontally.** Moreover, nullclines partition the phase plane into regions where x' and y' have one sign.

  Example: We sketch the nullclines and 'directions' for the system
$$\begin{aligned} x' &= x(-y+1-x), \\ y' &= y(x+1-y). \end{aligned} \tag{2}$$
The equilibria at $(0,0), (0,1)$ and $(1,0)$. The nullclines consist of two lines each, cutting the phase plane into several regions as shown below. We can assign a rough direction (up/down or left/right) in each region to get a sense of where orbits go in that region.

  The arrows suggest $(0,0)$ may be unstable and $(0,1)$ may be stable; $(1,0)$ is less clear. We will revisit this shortly with more tools to deduce its type.



1.2. **Invariant sets.** A set of points in the phase plane is called an **invariant set** if every solution that starts in that set stays inside for all t. The set is called **positively invariant** if it is true only as t increases and **negatively invariant** if it is true as t decreases.

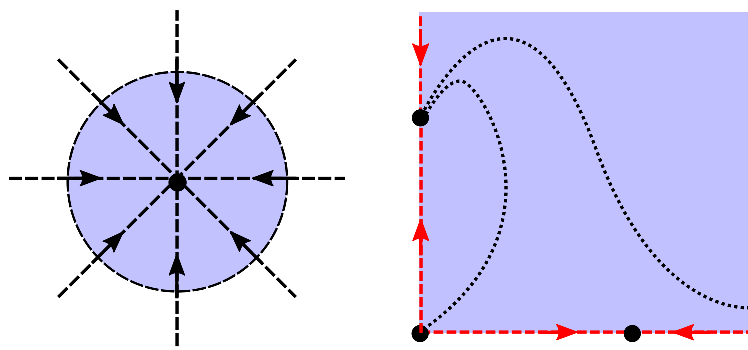
Every solution curve is an invariant set by definition. Positively invariant sets are useful because they 'trap' solutions: once a solution enters, it cannot leave (as t increases).

Example (trapping): For the star node

$$x' = -x,$$
$$y' = -y$$

Every disk centered at the origin is a positively invariant set (left figure below).

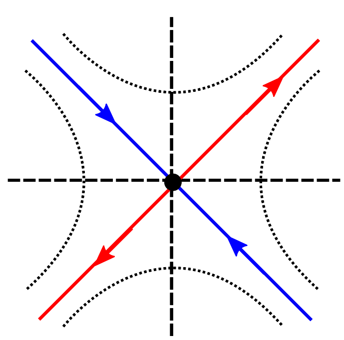
Example (barrier): For (2), the first quadrant $\{(x, y) : x, y > 0\}$ is an invariant set as the x and y axes are comprised of solution curves. They form a barrier solutions cannot cross.



Example (more barriers): Consider the linear saddle node

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

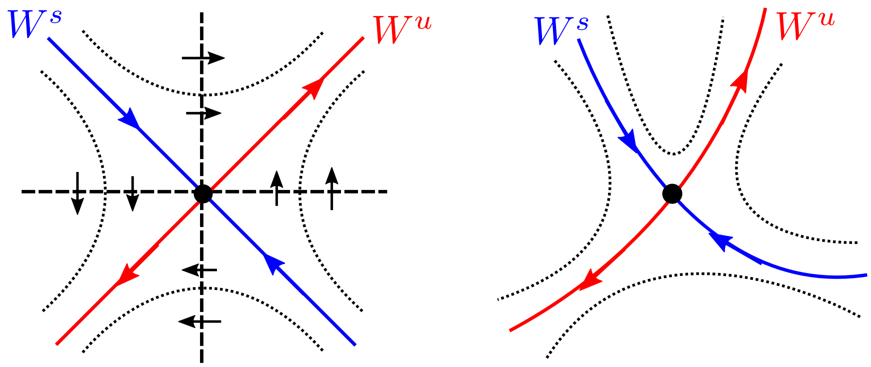
which has 'half-line' solutions along eigenvectors: $\mathbf{x} = e^t(1, 1)^T$ and $\mathbf{x} = e^{-t}(1, -1)$. These solutions are invariant sets. No solution can cross this pair of lines; the plane is cut into four wedges where solutions that start in that wedge stay there for all t.



1.3. **Stable/unstable manifolds.** Let $\mathbf{x}^*$ be an equilibrium point. We define the **stable manifold** $W^s(\mathbf{x}^*)$ and **unstable manifold** $W^u(\mathbf{x}^*)$ for $\mathbf{x}^*$ to be the sets

$$W^s(\mathbf{x}^*) = \{\mathbf{x}_0 : \text{the solution starting at } \mathbf{x}_0 \text{ converges to } \mathbf{x}^* \text{ as } t \to \infty\}$$
$$W^u(\mathbf{x}^*) = \{\mathbf{x}_0 : \text{the solution starting at } \mathbf{x}_0 \text{ converges to } \mathbf{x}^* \text{ as } t \to -\infty\}.$$

The stable manifold consists of solutions that are attracted to $\mathbf{x}^*$.
The unstable manifold consists of the solutions coming 'out' of the equilibrium.
Below are examples for a saddle node (left) and a sketch for a 'nonlinear' saddle node:



Example (LCC systems) For $\mathbf{x}' = A\mathbf{x}$, stable/unstable manifolds are easy to identify:

Stable/unstable point: For the stable node or spiral, the stable manifold of $(0,0)$ is $\mathbb{R}^2$ since every solution converges to $(0,0)$. The unstable manifold is empty. The opposite is true for unstable nodes and spirals (W^s empty, $W^u = \mathbb{R}^2$).

Center: No solution converges to $(0,0)$ for a linear center as $t \to \infty$ or $t \to -\infty$ (all solutions rotate around the equilibrium), so both W^s and W^u are empty!

Saddle node: The stable/unstable manifolds of a saddle node are each curves through $\mathbf{x}_0$; each segment is a solution curve. As a reminder, we looked at

$$x' = y, \quad y' = x$$

(see left figure above). The stable manifold is the line $y = -x$, comprised of the solutions

$$\mathbf{x} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{x} = -e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the unstable manifold is the line $y = x$; its two halves are the solution curves for

$$\mathbf{x} = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = -e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2. Qualitative analysis: Example

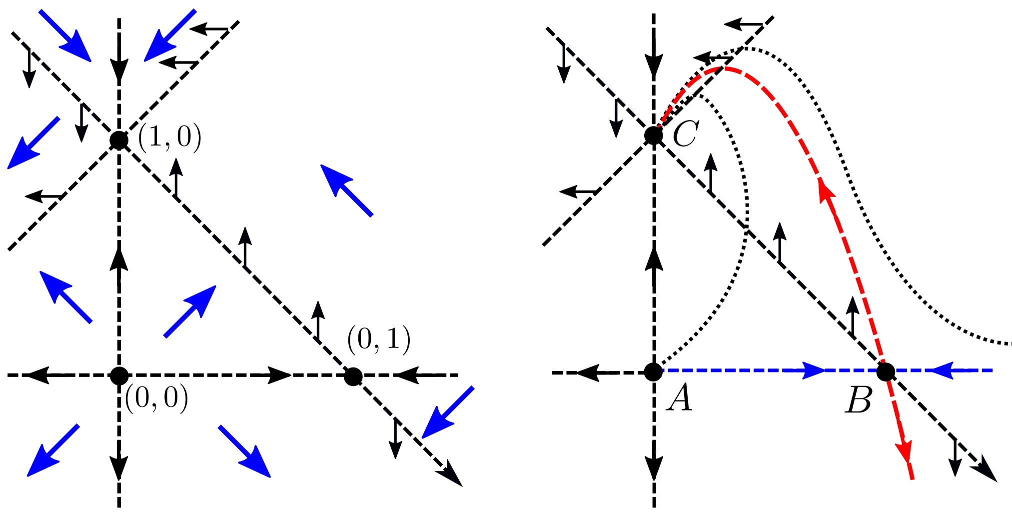
To get a sense of phase plane analysis, we consider the system (2) in more detail,

$$x' = x(-y + 1 - x), \quad y' = y(x + 1 - y),$$

focusing on the first quadrant $(x, y > 0)$. There are three equilibria:

$$A = (0, 0), \quad B = (1, 0), \quad C = (0, 1).$$

The nullclines are shown below. On the right is the phase plane with some solutions (from a computer plot, although we will see how to draw them by hand later).



First, observe that the x and y axes are invariant sets. If $y = 0$ then $y' = 0$, which means that solutions that start with $y = 0$ must have $y = 0$ for all t. In this case,

$$x' = x(1 - x)$$

so the dynamics on the line $y = 0$ (the x-axis; shown in blue) are the same as the familiar phase line for the logistic equation (if $x > 0$ then $x \to 1$ and if $x < 0$ then $x \to -\infty$).

Similarly, if $x = 0$ then $x' = 0$ and $y' = y(1 - y)$. By the non-intersection property, any solution that starts in the first quadrant must stay there, or else it would have to intersect the solutions on the x and y axis.

For solutions that start with $x, y > 0$, the plot suggests some features. The line in red above is the unstable manifold of B (the stable manifold is the x-axis). Notice that the half on the $y > 0$ side gets attracted to the equilibrium at C.

In fact, every solution that starts in the $x, y > 0$ quadrant gets attracted to C, swirling counter-clockwise (but not spiraling!). The plot suggests that A is 'unstable', C is 'stable' (at least on the $x > 0$ side) and B is a 'saddle node'.

Further analysis could be done for the other quadrants (left to you). At this point, it is clear that we need to update our definitions of stability (from phase lines) and 'types' of equilibria (saddle nodes etc.) for nonlinear systems.

3. Stability of equilibria

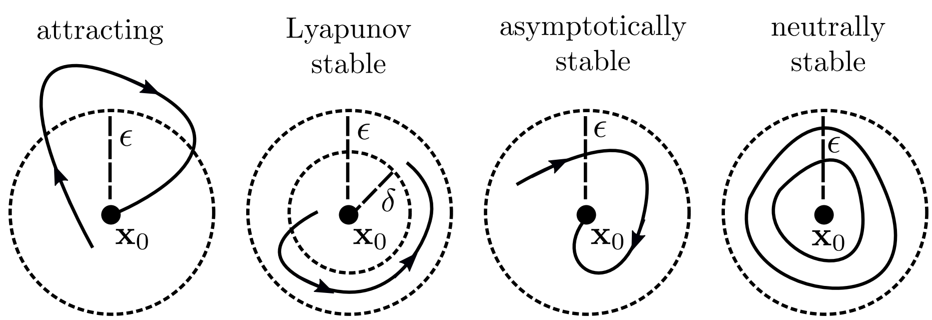
The fundamental definitions of stability for planar systems follow the same idea as in 1d.[2]

Definitions (stability): Let $\mathbf{x}^*$ be an equilibrium point for $\mathbf{x}' = F(\mathbf{x})$. As $t \to \infty$, we say

- $\mathbf{x}^*$ is **attracting** if all orbits near $\mathbf{x}^*$ converge to $\mathbf{x}^*$ as $t \to \infty$.
- $\mathbf{x}^*$ is **Lyapunov stable** if orbits near $\mathbf{x}^*$ stay near $\mathbf{x}^*$ as t increases.

The three main definitions are:

- $\mathbf{x}^*$ is **asymptotically stable** (a '**sink**') if it is Lyapunov stable and attracting (orbits nearby stay nearby for all time and converge to $\mathbf{x}^*$ as $t \to \infty$)
- $\mathbf{x}^*$ is **neutrally stable** if it is Lyapunov stable but not attracting (orbits that start close stay close, but don't always converge).
- $\mathbf{x}^*$ is **unstable** (a '**source**') if it is not Lyapunov stable or attracting (some solutions that start arbitrarily close will get repelled far away).



Technical note: The definitions can be made rigorous with some δ' and ϵ's (see textbook, Section 9.7). For instance, a point is Lyapunov stable if for each $\epsilon > 0$, there is a $\delta > 0$ such that when a solution starts within δ of $\mathbf{x}_0$, it stays within ϵ of $\mathbf{x}_0$ (shown in the figure).

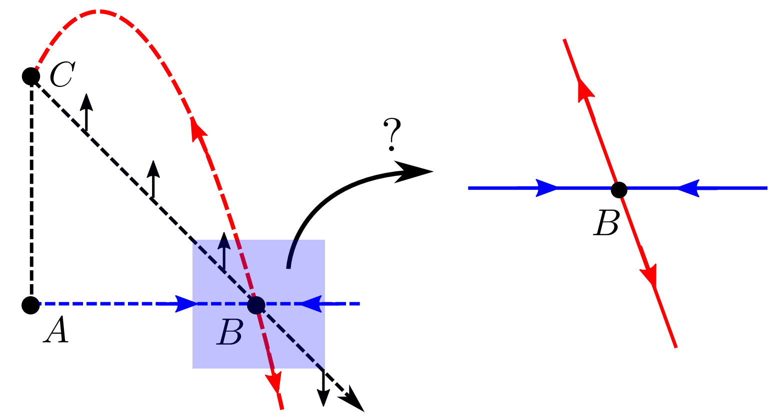
Remark: Neither one of 'Lyapunov stable' and 'attracting' implies the other. Lyapunov stable means orbits must stay close **for all** $t > 0$ (for attracting, only as $t \to \infty$). However, Lyapunov stable points do not require orbits converge to $\mathbf{x}^*$, while attracting points do.
Applying the definitions to LCC systems:

- **Saddle node:** Unstable. There are solutions that start arbitrarily close and end up far away (e.g. the unstable manifold).
- **Stable/unstable node (or spirals):** Stable nodes are asymptotically stable (solutions converge to the equilibrium as $t \to \infty$) and unstable nodes are unstable.
- **Center:** Lyapunov stable but not asymptotically stable. Every solution that starts close stays close for all t (orbits are closed) but none converge to $\mathbf{x}^*$.

[2]The textbook definitions are in Sec. 9.7. They use 'stable' instead of 'Lyapunov stable' and do not define 'attracting' . The definitions here arefrom Strogatz' *Nonlinear Dynamics and Chaos* Sec. 5.2.

4. Linearization and equilibria

Now we need a way to connect the behavior of equilibria for non-linear systems to the types we have encountered for linear (constant-coefficient) systems. The idea will be to zoom in close to the equilibrium; we will find that in some cases, it will then 'look like' the linear case.



Suppose $\mathbf{x}_0$ is an equilibrium point for $\mathbf{x}' = F(\mathbf{x})$. To approximate near $\mathbf{x}_0$, we **linearize** the ODE around $\mathbf{x}_0$.For a (smooth) function $F : \mathbb{R}^n \to \mathbb{R}^n$, Taylor's theorem states that

$$F(\mathbf{x}) = F(\mathbf{x}_0) + J(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \text{ higher order terms} \tag{3}$$

where J is the Jacobian

$$J = \left[\frac{\partial F}{\partial x_1} \;\middle|\; \frac{\partial F}{\partial x_2} \;\middle|\; \cdots \;\middle|\; \frac{\partial F}{\partial x_n} \right].$$

For the planar system with $F = (f, g)$ and $\mathbf{x} = (x, y)$, this is just

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

The equation (3) is the linear approximation to $F(\mathbf{x})$ around $\mathbf{x}_0$. We can then use this to approximate the ODE. Since $\mathbf{x}_0$ is an equilibrium point, $F(\mathbf{x}_0) = 0$ so

$$\mathbf{x}' = J(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + (\text{higher order terms in } \mathbf{x} - \mathbf{x}_0).$$

Now change variables to the position relative to $\mathbf{x}_0$:

$$\mathbf{w} = \mathbf{x} - \mathbf{x}_0.$$

Then $\mathbf{w}' = \mathbf{x}'$ and we get

$$\mathbf{w}' = J(\mathbf{x}_0)\mathbf{w} + (\text{higher order terms in } \mathbf{w}). \tag{4}$$

The equation (4) with the higher order terms dropped,

$$\mathbf{w}' = J(\mathbf{x}_0)\mathbf{w},$$

is called the **linearization** of the system at $\mathbf{x}_0$. It is an approximate system with constant coefficients for solutions when close to the equilibrium. We already know what the phase plane looks like for such a system (and even have the exact solution!). Moreover, we can deduce the stability of the equilibrium point $\mathbf{w} = 0$ just from the eigenvalues of $J(\mathbf{x}_0)$ (or use the trace-determinant phase diagram).

Example (1D): The ODE in 1D,

$$y' = y(1-y)$$

has a stable equilibrium point at $y_0 = 1$. Change coordinates to be relative to y_0:

$$w = y - y_0.$$

Now linearize by using Taylor's theorem on $f(y) = y(1-y)$ around $y_0 = 1$:

$$f(y) = f(y_0) + f'(y_0)(y - y_0) = 0 + (1 - 2y_0)(y - y_0) = -w.$$

Thus the linearized ODE for w is

$$w' = -w.$$

This ODE has a stable equilibrium at $w = 0$ and, more precisely,

$$w(t) = Ce^{-t}$$

which of course converges to zero exponentially fast as $t \to \infty$.

The two critical question are the following:

- Is it true that the phase plane for the nonlinear ODE near $\mathbf{x}_0$ looks like the phase plane for its linearization (and in what sense)?
- Does the stability predicted by the linearization predict the actual stability of $\mathbf{x}_0$?

It turns out that the answer to both questions is sometimes yes, sometimes no. Some examples will illustrate the point.

4.1. **Linearization example (it works... sometimes).** We can use linearization to verify the observations about the example (2) at $A = (0,0)$ and $B = (1,0)$. Recall the system is

$$x' = x(-y + 1 - x), \quad y' = y(x + 1 - y).$$

The Jacobian is

$$J = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 - y - 2x & -x \\ y & 1 + x - 2y \end{bmatrix}.$$

First, we look at the origin, which was observed to be unstable. We have

$$J\Big|_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the linearized system $\mathbf{x}' = J\mathbf{x}$ is an unstable star node. In particular, A is unstable. It is indeed true that the actual phase plane looks like the star node near A.

Now consider B, which looked like a saddle node. We have

$$J\Big|_B = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

The trace/determinant are $\tau = 1$ and $D = -2$. Since $D < 0$, the linearized system

$$\mathbf{w}' = J\mathbf{w} \quad \text{for } \mathbf{w} = \mathbf{x} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a saddle node. The eigenvalues/eigenvectors are

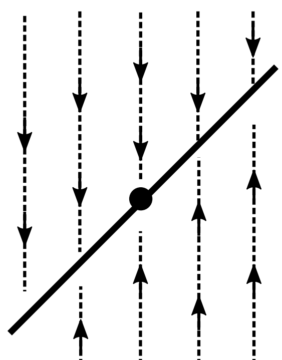
$$\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Notice that the eigenvector $\mathbf{v}_1$ for the negative eigenvalue is horizontal. In fact, the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ have the same slope as the stable/unstable manifolds for the real equilibrium point! Thus the eigenvectors of the linearization tell us, locally, what the stable/unstable manifolds of the saddle point look like.

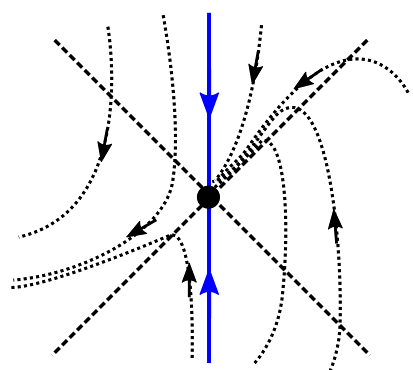
Finally, at $C = (0, 1)$ we have

$$J\Big|_C = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are $\lambda = 0$ and $\lambda = -1$. The phase plane for the linearization is one of the degenerate cases, where solutions lie on vertical lines converging to the line $x = y - 1$. But this does not match what the actual phase plane looks like! A sketch is shown below. Note that in the real case, solutions that start with $x > 0$ converge to $(0, 1)$, while the linearization predicts a line of equilibria for $x = y - 1$.



Linearization at $(0, 1)$



Actual phase plane at $(0, 1)$

so the linearization does **not** correctly predict the behavior. For the right approach and the details, see the homework.

5. When linearization works

It turns out that eigenvalues with zero real part are the culprit for spoiling the linearization method. Otherwise, the method works. One important result[3] (not easy to prove!) is the following theorem:

[3] This is an informal version of two related theorems, the *stable manifold theorem* and the *Hartman-Grobman* theorem. The details (and the difference between the two) are beyond the scope of the course.

Theorem: Let $\mathbf{x}_0$ be an equilibrium point for the planar system $\mathbf{x}' = F(\mathbf{x})$. If the eigenvalues λ_1, λ_2 of the Jacobian at $\mathbf{x}_0$ have

$$\text{Re}(\lambda_1) \neq 0, \quad \text{Re}(\lambda_2) \neq 0$$

then the linearization accurately predicts the behavior near $\mathbf{x}_0$ (in that the phase planes 'look similar' near $\mathbf{x}_0$). In particular, this holds for stable/unstable nodes, saddle nodes and spirals but not centers.

For a saddle node, the stable/unstable manifolds connect to $\mathbf{x}_0$ tangent to the stable/unstable manifolds for the linearization (i.e. in the directions of the eigenvectors).

Thus, for these types of equilibria, we can use the linearization to sketch the phase plane around any equilibrium with good accuracy. Note: 'looks similar' allows for some distortion between the linearized and actual phase plane; this can be significant when $\lambda_1 = \lambda_2$. At the very least, the **stability** of the equilibrium is always correctly predicted.

5.1. **More on the not so nice cases.** For the cases where the theorem does not apply, it is much more difficult to deduce the local behavior; the theory is beyond the scope of the course. However, a few examples will illustrate the idea.

As we have just seen, the linearization **does not always match the behavior near the equilibrium point**. This discrepancy happens, in particular, when there is an eigenvalue with zero real part. Because the leading order term does not decay, it is sensitive to influence from the higher order terms. For instance, consider in 1D

$$y' = -\epsilon y + y^3.$$

If $\epsilon > 0$, the linearization around the equilibrium at $y = 0$ is

$$y' = -\epsilon y$$

which is a stable point. Indeed, $y = 0$ is stable when $\epsilon > 0$.

However, if $\epsilon = 0$ then $y' = y^3$ and the linearization is

$$y' = 0.$$

The linearization predicts neutral stability - nothing moves close or far away. However, the next term (the y^3 term) makes the equilibrium unstable.

This phenomenon often happens with centers in planar systems. Consider the following system:[4]:

$$\begin{aligned} x' &= -y + ax(x^2 + y^2) \\ y' &= x + ay(x^2 + y^2) \end{aligned}$$

where a is a constant. There is an equilibrium point at $(x, y) = (0, 0)$. Note that when computing the partial derivatives at $(0, 0)$, all the parts from the second term will be zero,

[4]Example borrowed from Strogatz, *Nonlinear Dynamics and Chaos*, Chapter 6

e.g.

$$\frac{\partial}{\partial y}(-y + ax(x^2+y^2)) = -1 + 2axy = -1 \text{ at } (0,0).$$

The Jacobian at $(0,0)$ is then

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so the linearization predicts a center (circles!). However, we can analyze this system directly, by using polar coordinates. Let

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We have that

$$r^2 = x^2 + y^2$$

so (for a solution $(r(t), \theta(t))$ in polar coordinates and $(x(t), y(t))$ in rectangular coordinates)

$$rr' = xx' + yy'.$$

Plugging in x' and y' from the ODE, we get

$$rr' = x(-y + axr^2) + y(x + ayr^2) = ar^2(x^2+y^2) = ar^4$$

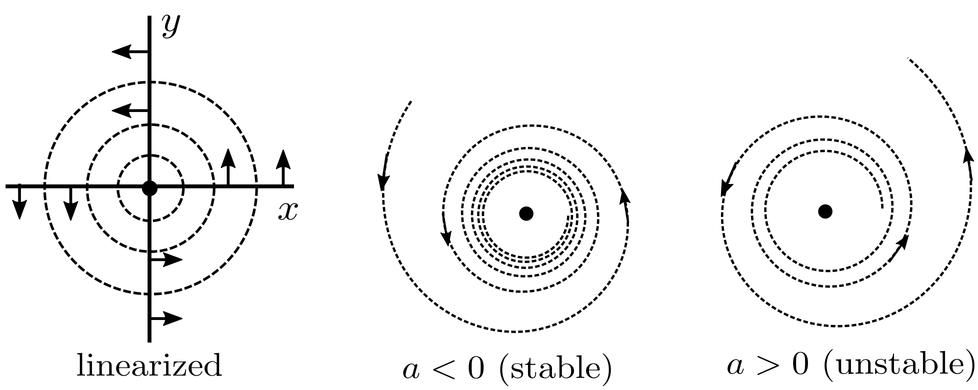
so

$$r' = ar^3.$$

It is not too hard to show (see homework) that

$$\theta' = (xy' - yx')/r^2 = 1.$$

Thus the solution rotates counter-clockwise (at an angular speed of 1). We see that $r = 0$ is an equilibrium point. Solutions decrease to $r = 0$ (stable) if $a < 0$ and increase away from $r = 0$ (unstable) if $a > 0$. In the phase plane, this means solutions spiral inward if $a < 0$ and outward if $a > 0$.



Thus $(0,0)$ is in fact a stable or unstable spiral, and not a center! The linearization is incorrect. The problem is easy to see in polar coordinates. Linearizing, we would get

$$r' = ar^3 \approx 0, \quad \theta' = 1.$$

But we cannot discard the ar^3 term, because it is the one responsible for the growth/decay of r. The linearization does not see this term and makes the wrong prediction.

We can gain some intuition by looking at the phase diagram (the trace-determinant plane) characterizing the linear cases. When on the boundary between cases, the system is sensitive to small adjustments. Non-linear terms that are normally small **can push the system into one case or another.** Centers can become stable/unstable spirals.

A stable star node lies on the boundary between stable nodes and stable spirals. Non-linear terms canturn them into stable spirals. However, they cannot become unstable; the linearization will still predict the stability correctly. The linearization will be 'correct' in the sense that the actual phase plane is a slightly twisted version of the star node.

Similarly, if the linearization predicts a degenerate stable node, the actual behavior can be different but it will stay stable.

6. An example: phase portraits

Now we have enough enough tools to sketch the phase portrait for a system and describe its behavior. Let's consider a population of rabbits and squirrels, $x(t)$ and $y(t)$, both of which are competing for the same space.[5]

Assume that, in isolation, each species grows up to a certain carrying capacity, according to the logistic equation.

Further assume that the due to competition (e.g. from competing over the same food source), the growth rate is reduced, proportional to the population of both species.

The model equations will look like

$$x' = r_1 x(K_1 - x - b_1 y), \quad y' = r_2(K_2 - b_2 x - y),$$

which is an example of the **Lotka-Volterra model** for population growth. The phase portrait depends on the parameters. As an example, we will choose nice ones and study

$$\begin{aligned} x' &= x(3 - x - 2y) \\ y' &= y(2 - x - y). \end{aligned}$$

Since x, y are populations, we only care about the first quadrant of the phase plane.

We want to answer the usual question: what happens to the population, and how does it depend on the initial values? Can they coexist? Do they reach an equilibrium?

We will go step by step to build a phase portrait[6] that captures all the qualitative features. The steps here are not a rigorous procedure.

(First) Step: Find the equilibria/nullclines: The equilibria occur where

$$(x = 0 \text{ or } x + 2y = 3) \quad \text{and} \quad (y = 0 \text{ or } x + y = 2).$$

Thus there are four cases. The equilibria are

$$E = (0, 0), \quad R = (3, 0), \quad S = (0, 2), \quad C = (1, 1).$$

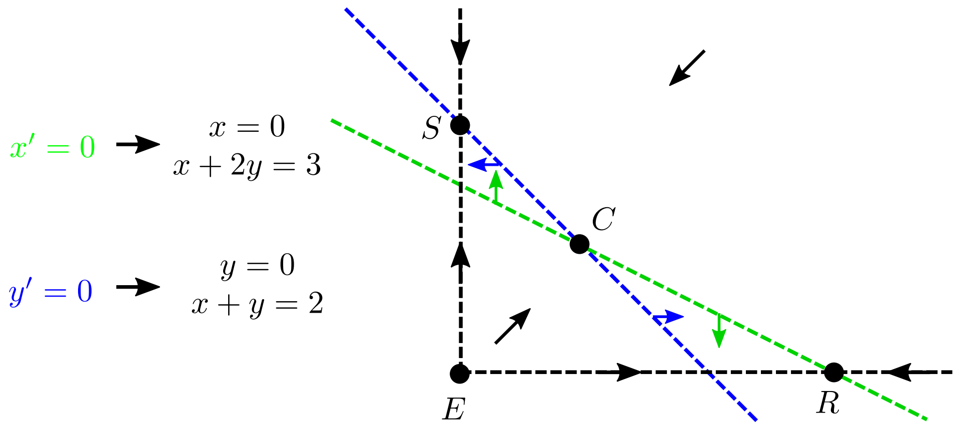
[5]Example adapted from Strogatz, *Nonlinear Dynamics and Chaos.*

[6]A *phase portrait* is a picture of the phase plane with all the important features and enough solution curves sketched in that you can see what all solutions will do at a glance.

Each equilibrium represents something for the system:

- E: extinction; the populations have both died out
- R: only rabbits remain
- S: only squirrels remain
- C: coexistence; both populations exist together.

The nullclines with directions are shown below:



Step: Identify useful invariant sets and boundaries: Note that if $x = 0$ then $x' = 0$. Thus a solution that starts on $x = 0$ stays there forever. This makes sense; if there are no rabbits then the squirrel population is governed by

$$y' = y(2 - y).$$

Similarly, if $y = 0$ then $y' = 0$ and $x' = x(3 - x)$. It follows that solutions curves in the first quadrant stay there forever (so $\{(x, y) : x, y > 0\}$ is an **invariant set**). This fact is useful because we only care about this quadrant anyway, so we do not need to worry about solutions leaving the region where they make physical sense.

Step: Determine linearizations: Now let's go through and determine the linearization at each equilibrium point. Phase portraits for the cases are **shown at the end**. The Jacobian is

$$A = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}.$$

For $E = (0, 0)$: The Jacobian is

$$A\Big|_E = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

which is an unstable node. The fast/slow directions are $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$ (solutions want to be tangent to $\mathbf{v}_1$ as $t \to \infty$ since this is the e^{3t} term).

For $R = (3, 0)$:

$$A\Big|_R = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}, \quad \lambda_1 = -3, \ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = -1, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

A stable node; the fast direction is $\mathbf{v}_1$ direction and the slow direction is $\mathbf{v}_2$ (solutions want to be tangent to $\mathbf{v}_2$ as $t \to \infty$ because e^{-t} will be the larger than e^{-3t}).

For $S = (0, 2)$:

$$A\Big|_S = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}, \quad \lambda_1 = -1,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \lambda_2 = -2,\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A stable node; the slow direction is $\mathbf{v}_1$.

For $C = (1, 1)$:

$$A\Big|_C = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}.$$

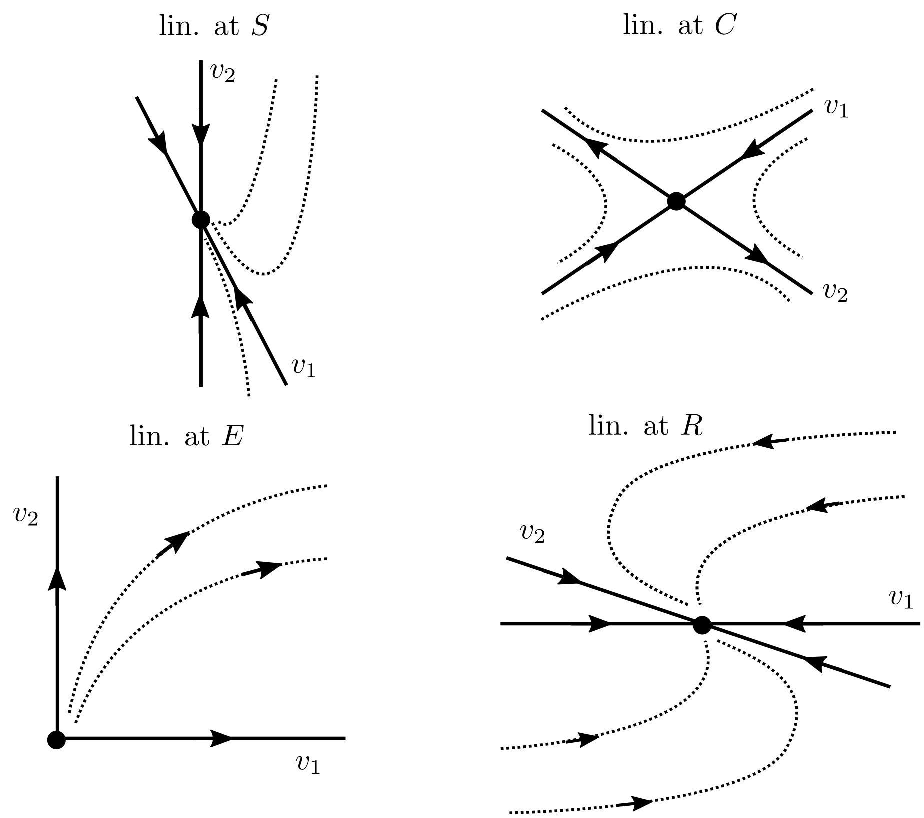
The trace/determinant are $\tau = -1$ and $D = -1$; since $D < 0$ this is a saddle node. Computing the eigenvalues/vectors explicitly:

$$\lambda_1 = -1 - \sqrt{2},\ \mathbf{v}_1 = \begin{bmatrix} -2 \\ -\sqrt{2} \end{bmatrix}, \quad \lambda_2 = -1 + \sqrt{2},\ \mathbf{v}_2 = \begin{bmatrix} -2 \\ \sqrt{2} \end{bmatrix}$$

Note that it is useful to recall the trick that

$$\lambda \text{is an eigenvalue of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies v = \begin{bmatrix} b \\ \lambda - a \end{bmatrix} \text{ is an eigenvector.}$$

The stable manifold is in the $\mathbf{v}_1$ direction ($\mathbf{v}_1 \approx (2, 1.4)$, scaling it by -1 to get rid of negative signs) and the unstable manifold is in the $\mathbf{v}_2$ direction ($\mathbf{v}_2 \approx (-2, 1.4)$).



Step: Draw essential solution curves: First, we sketch the linearized phase portraits around their respective equilibrium points in the 'good cases' where they are correct predictions. **Note that in all cases**, there is no eigenvalue with zero real part, so all of them are correct[7].

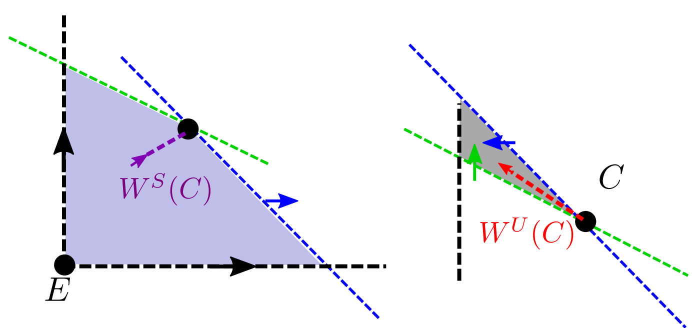
Now follow stable/unstable manifolds from saddle points. There are four curves to look at: the two halves of the stable/unstable manifolds of C. We know what they look like near C from the linearization.

For the stable manifold, see the figure below. We follow it backwards in time ($t \to -\infty$) out of C. We claim that as $t \to -\infty$, it goes to the origin (P). To argue this (informally), let Ω be the region next to the origin (shaded blue in the figure). It is bounded on the top/right by nullclines, and the vector field points out of Ω for both. The other two sides are parts of the x/y axes which we know are solution curves. Thus any solution curve that leaves Ω must do so as t increases, through the nullclines. This means that the stable manifold is trapped in Ω.

Since the solution curve must have $x', y' > 0$, it cannot do anything strange, so it must originate from P (i.e. converge to P as $t \to -\infty$.

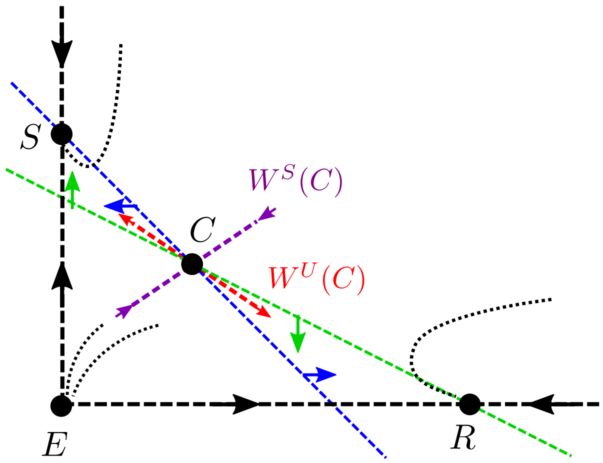
For the other half of the stable manifold, as $t \to -\infty$ it diverges with x or y to ∞ (probably both, but we do not have an argument for this).

For the unstable manifolds, the left half leaves C and converges to S; the right half leaves C and converges to R. One can make similar arguments to the one we made for C; a picture is shown below (exercise: explain this in detail).



Step: Draw other curves: Now fill out the phase portrait by extending the other local curves we sketched to the whole phase plane. As a reminder, we showed that the first quadrant was invariant, so it is safe to plot only this quadrant (nothing enters from outside or leaves).

[7]In the 'bad cases', it is not correct to do this; we would need to use other methods to determine the local behavior



(Last) Step: Conclude something: Presumably, the phase portrait is being drawn for some purpose. Here, we wanted to know: what is the fate of the population and how does it depend on the initial conditions The key observation is that the stable manifold of C cuts the first quadrant into two pieces. One has R on its boundary, the other has S.

The phase portrait then reveals the answer: If a solution curve starts above the stable manifold $W^S(C)$, it will converge to R. If a solution curve starts below $W^S(C)$, then it will converge to S.

Coexistence is possible, but only if the populations lie on the stable manifold of C; then they will converge to the values at C. However, if the population is perturbed slightly, then depending on whether it is pushed into Ω_R or Ω_S, one species will 'win' and the other will die out. We conclude, therefore, that in the absence of any other stabilizing effects, coexistence is not really possible.

There is no (non-trivial) initial condition where both species die out (the origin is unstable).