MATH 356 LECTURE NOTES
INTRO TO (QUALITATIVE) DYNAMICS
PHASE PLANES IN 2D: LCC CASE

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Topics covered

- Autonomous equations in 2d (general)
  - Introduction; definitions
  - The phase plane
- LCC systems in 2d
  - Categorization of equilibria
  - Saddles, nodes, spirals; sketching the phase plane
  - Trace-determinant plane, degenerate cases

1. Planar autonomous systems: definitions

We now increase the dimension by 1 and study the planar (2D) system

$$x' = f(x, y), \quad y' = g(x, y).$$

This is an autonomous first-order system in $\mathbb{R}^2$ in the form

$$\mathbf{x}' = F(\mathbf{x}) \tag{1}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ (given by $F = (f, g)$) and $\mathbf{x} = (x, y)$.

Equilibria: As before, a point $\mathbf{x}^*$ is an **equilibrium point** (or fixed point) if

$$F(\mathbf{x}^*) = 0.$$

This is true if and only if $\mathbf{x}(t) = \mathbf{x}^*$ is a constant solution.

Phase plane: The **phase plane** is the diagram showing solutions on (x, y) plane. Solutions $(t, x(t), y(t))$ are projected onto the (x, y) plane, so solutions move in the plane as t changes. Solution curves follow the vector field (x', y') (or $\mathbf{F}$). As an example, for the system

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x},$$

the solution curves are circles centered at the origin (Figure 1), with a single equilibrium point at $(0, 0)$. The solutions themselves rotate clockwise along the solution curve.

Orbits: A solution curve in the phase plane starting at (x_0, y_0) is called an **orbit** (the orbit of that point) or a **trajectory** of the system (or **solution curve** or solution). An orbit that forms a closed curve is called a **closed orbit**.

Nullclines: The sets where $x' = 0$ and $y' = 0$ are called the x- and y- **nullclines**. Solution curves must cross the x-nullcline in the y-direction and vice versa. Plotting the vector field on the nullclines helps to see where solutions can go.

Moreover, as shown in Figure 1, the nullclines split the phase plane into regions where x' and y' have certain signs, since x' and y' can only change sign by crossing a nullcline. Labeling the regions with directions (up/down/left/right) helps to guide sketches and analysis.

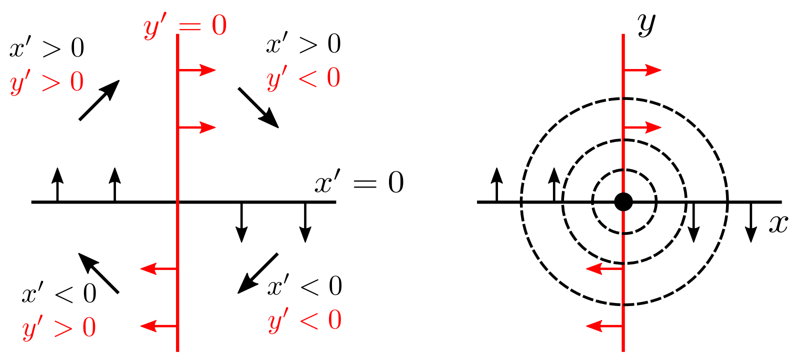


FIGURE 1. Left: Nullclines (y-nullcline in red) and regions where x', y' have a given sign for $x' = y, y' = -x$. Right: Some solution curves (circles).

2. Linear planar systems

We shall now analyze the phase plane in detail for

$$\mathbf{x}' = A\mathbf{x}$$

where A is a constant 2×2 matrix.

Notation for this section: The system, in scalar form, looks like

$$x' = a_{11}x + a_{12}y, \quad y' = a_{21}x + a_{22}y$$

with $\mathbf{x} = (x, y)$. We will switch freely between system and scalar notation. Assume here that A is invertible (see HW for the other cases).

The exact solutions can be used to give an exhaustive characterization of the types of phase planes that can arise. Since A is invertible, there is a unique equilibrium point at $(0, 0)$; we will classify the equilibria in terms of the eigenvalues of A.

2.1. **Stable node.** Let

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}.$$

The eigenvalues/vectors are $\lambda_1 = -4$ and $\lambda_1 = -2$ with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The solution is

$$\mathbf{x} = c_1 e^{-2t}\mathbf{v}_1 + c_2 e^{-4t}\mathbf{v}_2.$$

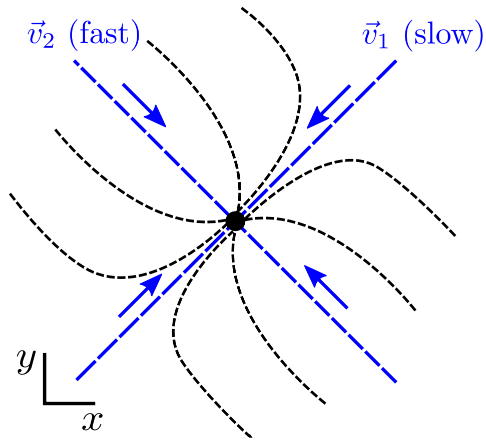
All solutions converge to $(0,0)$. Note that $\mathbf{v}_1$ is the 'slow' direction and $\mathbf{v}_2$ is the 'fast' direction (decreases faster as $t \to \infty$ and increases faster as $t \to -\infty$). As $\to \infty$, we have

$$\mathbf{x} \approx c_2 e^{-2t} \mathbf{v}_1.$$

That is, for large t, all solutions tend to become parallel to $\mathbf{v}_1$. Similarly, as $t \to -\infty$,

$$\mathbf{x} \approx c_1 e^{-4t} \mathbf{v}_2$$

so solutions will tend to be parallel to $\mathbf{v}_2$. Here is a picture:



The black dashed lines are solutions (that all converge to the origin). This type of equilibrium is called a **stable node**, where all solution curves converge to $(0,0)$ and the eigenvalues are real. One term is dominant in each limit $t \to \infty$ and $t \to -\infty$, and the direction of the solution in each limit is determined by the eigenvectors. The opposite case is called an **unstable node** (same picture, solutions just go the other way along the solution curves).

Asymptotes: of the lines $x = y$ and $x = -y$ separated by the origin are solution curves (corresponding to initial conditions with $c_2 = 0$ and $c_1 = 0$, respectively). These curves are special because they are simply pieces of lines in the phase plane.

Note on sketching: A more precise picture could be drawn by also sketching in the nullclines $y = 3x$ (for $x' = 0$) and $x = 3y$ (for $y' = 0$). The eigenvectors are critical for the sketch; the nullclines just help get the right shape in-between.

2.2. **Saddle node.** An equilibrium can have both stable and unstable components. Consider

$$x' = y, \;\; y = x, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The x and y nullclines are $y = 0$ and $x = 0$, which will be useful in drawing the phase plane. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

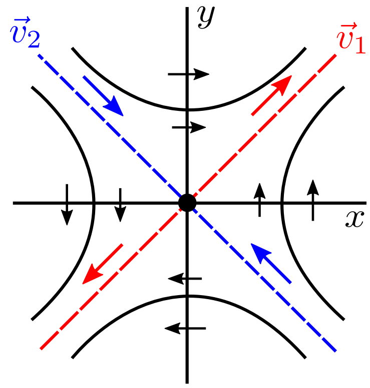
The solution is

$$\mathbf{x} = c_1 e^{t} \mathbf{v}_1 + c_2 e^{-t} \mathbf{v}_2.$$

Observe that $\mathbf{x} = c_1 e^t \mathbf{v}_1$ is a solution; the solution curve lies on the line through the origin in the direction $\mathbf{v}_1$. Solutions on this line move away from the origin along $\pm \mathbf{v}_1$. This line is called the 'unstable manifold' (to be revisited); it is analogous to a phase line with an unstable point.

Another (linearly independent) solution is $\mathbf{x} = c_2 e^t \mathbf{v}_2$. The situation is the same but solutions instead converge to the origin. This is called the 'stable manifold'.

The vector field on the nullclines (black arrows), the lines along $\mathbf{v}_1, \mathbf{v}_2$ through the origin and some solutions are shown below.



For other initial conditions, the same analysis as the previous case applies:

$$\mathbf{x} \approx c_1 e^t \mathbf{v}_1 \text{ as } t \to \infty, \quad \mathbf{x} \approx c_2 e^{-t} \mathbf{v}_2 \text{ as } t \to -\infty.$$

But now, in either case, $\mathbf{x}$ is diverging. So, if we follow the solution forwards, it will tend towards the unstable manifold, diverging to ∞, and if we follow it backwards, it will tend towards the stable manifold, also diverging.

Note that only solutions on the stable manifold (the line $y = -x$) will actually converge to the origin. All others will get deflected and end up diverging along the asymptote $y = x$.

Additional note: The solution curves are just hyperbolas with asymptotes at $y = \pm x$. This makes sense since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = x/y$$

which is a separable equation whose solutions are $x^2 - y^2 = C$. This type of equilibrium, with one unstable and one stable direction, is called a **saddle node**.

2.3. **Center.** Revisit the equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In component form, this is $x' = y$ and $y' = -x$. The solutions satisfy $x^2 + y^2 = C$ since

$$\frac{d}{dt}\left(x(t)^2 + y(t)^2\right) = 2xx' + 2yy' = 2xy + 2y(-x) = 0.$$

So the solution curves are circles, and the solutions rotate around the circle. Indeed, the eigenvalues are $\lambda \pm i$ and the eigenvectors are $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 i$ so

$$\mathbf{x}(t) = c_1(\cos t\mathbf{e}_1 - \sin t\mathbf{e}_2) + c_2(\sin t\mathbf{e}_1 + \cos t\mathbf{e}_2).$$

Here $\mathbf{e}_1 = (1,0)^T$ and $\mathbf{e}_2 = (0,1)^T$. One solution ($c_2 = 0$) is

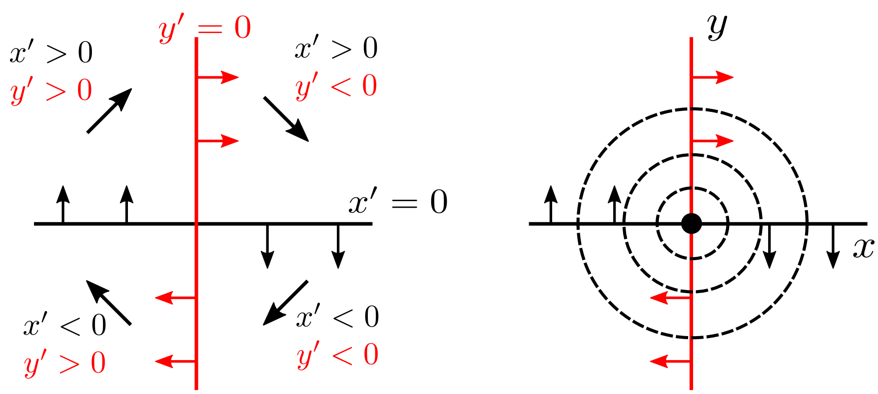
$$x = c_1 \cos t, \quad y = -c_1 \sin t$$

which is just clockwise rotation around the origin (radius c_1). This is enough to generate all the solution curves (all the other solutions with $c_2 \neq 0$ be the same, just shifted in t).

Such an equilibrium point is called a **center**: solutions rotate around but do not converge to $(0,0)$. Note that we can figure out the direction by testing one point. At $\mathbf{x} = (1,0)$, the vector field is

$$A\mathbf{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

which points down, so the solution must go clockwise. Plotting the nullclines and the vector field on the nullclines shows the rotation as well.



A messier example: Centers do not have to be circles. Consider

$$A = \begin{bmatrix} -1 & -1 \\ 5 & 1 \end{bmatrix}.$$

which has eigenvalues $\lambda = \pm 2i$. The eigenvectors are $\mathbf{v}_1 \pm \mathbf{v}_2 i$ where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Note that the complex eigenvectors can be scaled by a complex number, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are *not* unique (see comment at the end of the example). A complex solution is

$$\mathbf{x} = e^{3it}(\mathbf{v}_1 + \mathbf{v}_2 i).$$

Taking real/imaginary parts we get

$$\mathbf{x} = c_1(\cos 2t\mathbf{v}_1 - \sin 2t\mathbf{v}_2) + c_2(\sin 2t\mathbf{v}_1 + \cos 2t\mathbf{v}_2).$$

Grouping by the eigenvectors,

$$\mathbf{x} = (c_1 \cos 2t + c_2 \sin 2t)\mathbf{v}_1 + (-c_1 \sin 2t + c_2 \cos 2t)\mathbf{v}_2.$$

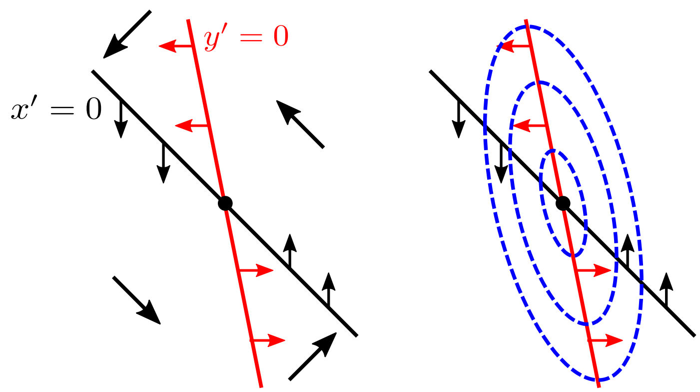
Setting $c_2 = 0$ (like the previous case, we don't need the c_2 terms to draw the solution curves) we get solutions

$$\mathbf{x} = c_1 \cos 2t\mathbf{v}_1 - c_1 \sin 2t\mathbf{v}_2,$$

which are ellipses. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ do not tell us too much about the solution curves because they are not orthogonal (they are **not** the major/minor axes of the ellipse).

The direction of rotation is easy to find, however: just test a point. For instance, when $\mathbf{x} = (1, 0)$ we have $A\mathbf{x} = (-1, 5)$ so the rotation is counter-clockwise. The tilt of the ellipse can be estimated in the sketch using the x and y nullclines,

$$y = -x/2 \quad \text{and} \quad y = -5x.$$



Through this relatively simple analysis, we can get a good but not perfect picture of the shape of the ellipse. **Optional note:** In general, nullclines and ellipse axes are not related. With much more work, we can pick $\mathbf{v}_1$ and $\mathbf{v}_2$ (the real/imaginary parts of the eigenvector) to be orthogonal, in which case they will be the directions of the major/minor axes.

2.4. **Spiral.** If the λ's are complex with a real part, solutions will also grow or decay. Let

$$A = \begin{bmatrix} -1 & -2 \\ 10 & 3 \end{bmatrix}.$$

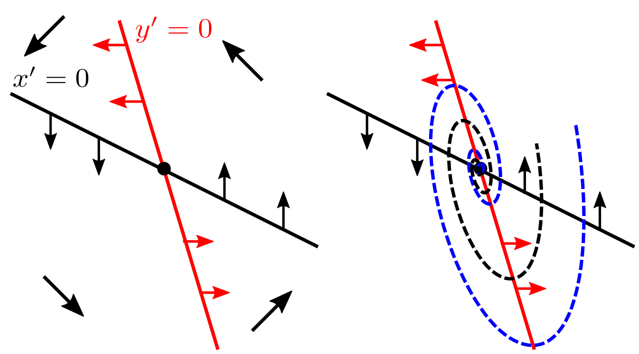
which has eigenvalues $\lambda = 1 \pm 4i$ and eigenvectors are $\mathbf{v}_1 \pm \mathbf{v}_2 i$ where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Taking real/imaginary parts of a solution (as in the previous case):

$$\mathbf{x} = c_1 e^t (\cos 4t \mathbf{v}_1 - \sin 4t \mathbf{v}_2) + c_2 e^t (\sin 4t \mathbf{v}_1 + \cos 4t \mathbf{v}_2).$$

Thus solutions spiral inwards. The spiral is the most difficult case to draw by hand. Again, checking the vector field on some axis (e.g. at $(0, 1)$) tells us the direction is counterclockwise. The nullclines $y = -x/2$ and $y = -10x/3$ give some indication of the shape of the spiral.



This equilibrium point is called a **unstable spiral** because it spirals outward.

2.5. **Degenerate cases:** Detailed after the next section.

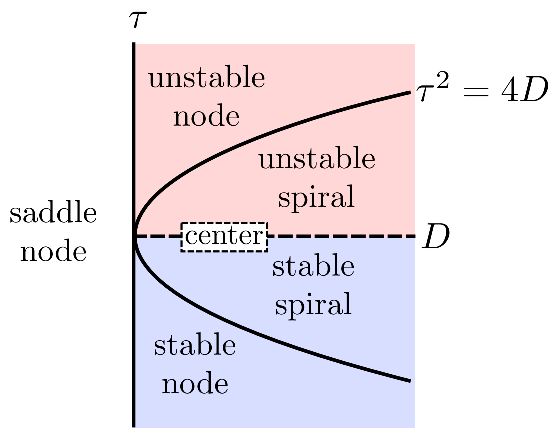


FIGURE 2. The 'trace-determinant' plane characterizing equilibria for the LCC system (2).

3. PHASE DIAGRAM

Now we can map out what type of equilibrium appears given a system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2}$$

The components a, b, c, d are not convenient to work with. Instead, recall that the **trace** $\text{tr}(A)$ of a square matrix is the sum of its main diagonal elements. Define

$$\tau = \text{tr}(A) = a + d, \quad D = \det(A) = ad - bc.$$

The characteristic polynomial is

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - \tau\lambda + D.$$

Thus the eigenvalues are

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4D}}{2}. \tag{3}$$

The values of (τ, D) therefore determine the equilibrium type. We illustrate this by plotting a 'phase diagram', marking in the (τ, D) plane the set of (τ, D) corresponding to each case.

The main features of the are shown below. The parabola $\tau^2 = 4D$ separates cases with complex roots (spirals, centers) from those without (everything else).

- If $D < 0$ then the eigenvalues are real with opposite signs (saddle node).
- If $\tau^2 < 4D$ then the λ's are complex with a real part (a spiral; unstable if $\tau > 0$ and stable if $\tau < 0$).
- If $\tau = 0$ and $D > 0$ then the λ's are purely imaginary (a center)
- If $D > 0$ and $\tau^2 > 4D$: λ's are real with the same sign (a node; stable if $\tau > 0$)

Checking these main cases in detail is left to you (one example is shown below). Some degenerate cases are left out, which lie on the boundary between types (see next section).

Example (characterizing spirals): For example, to identify unstable spirals, we need eigenvalues that are complex with a positive real part. Note that, from (3),

$$\lambda \text{ is complex} \iff \tau^2 < 4D.$$

If λ is complex then

$$\lambda = \frac{\tau}{2} \pm \frac{i}{2}\sqrt{4D - \tau^2}$$

so $\mathrm{Re}(\lambda) > 0$ if and only if $\tau > 0$. Thus unstable spirals lie above $\tau = 0$ and below $\tau = \sqrt{4D}$ on the trace-determinant plane (as depicted on the diagram).

3.1. **Degenerate cases.** There are several degenerate cases to consider. They lie on the boundary between regions on the trace-determinant plane. A brief treatment:

Star node: λ is repeated, but has a two LI eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. In this case the solution is

$$\mathbf{y}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2).$$

Note that for any vector $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we have $A\mathbf{v} = \lambda\mathbf{v}$, so $A = \lambda I$ (a multiple of the identity). Thus star nodes are exactly the (uninteresting) systems

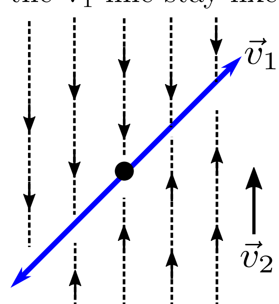
$$x' = \lambda x, \quad y' = \lambda y.$$

Star nodes lie on the boundary $\tau^2 = 4D$ between nodes and spirals. They are not quite 'twisted' like a spiral, and do not quite have two distinct fast/slow directions like a node (all directions are equally fast).

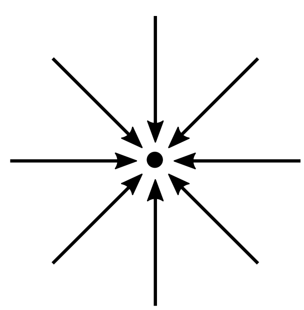
Line of equilibria: $\lambda_1 = 0$ and $\lambda_2 \neq 0$. This is the case where the equilibrium point is **not unique**, since the solution is

$$\mathbf{y}(t) = c_1\mathbf{v}_1 + c_2 e^{\lambda t}\mathbf{v}_2$$

There is a line of equilibria at $c_1\mathbf{v}_1$ (for any c_1). Solutions not on $\mathbf{v}_1$ either converge along $\mathbf{v}_2$ ($\lambda < 0$) towards the line or diverge. Note that no equilibrium point is asymptotically stable, since nearby points on the $\mathbf{v}_1$ line stay fixed.



($\lambda_1 = 0,\ \lambda_2 < 0$)



star ($\lambda < 0$)

- **Degenerate node:** See homework; the interesting degenerate case. One eigenvalue λ, repeated with one eigenvector $\mathbf{v}_1$. In this case, the solution is

  $$\mathbf{y}(t) = e^{\lambda t}(c_1(\mathbf{v}_2 + t\mathbf{v}_1) + c_2\mathbf{v}_2)$$

  where $\mathbf{v}_2$ is the generalized eigenvector.