**Reading:** Chapter 13.4 and 13.5.

**Grading note:** Friday is a hard deadline for this homework. The problems to be turned in and graded are starred. Problems P3 and P4 are relevant to the exam and should also be considered part of studying for the test.

**Problems:** Complete the following problems:

- Chapter 13.4: *7, *8
- Non-book problems below

---

**P1.** Solve Laplace’s equation outside a semi-circle of radius 2 with homogeneous Dirichlet BCs on the flat parts and Neumann BCs specified on the curved parts:

\[
\begin{align*}
&u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0, \quad r \in (2, \infty), \quad \theta \in [0, \pi] \\
&u(r, 0) = u(r, \pi) = 0, \quad r \in [0, 2] \\
&u_r(2, \theta) = \cos \theta, \quad \theta \in [0, \pi]
\end{align*}
\]

a) Use separation of variables to find all separated solutions. Impose the condition that

\[
\lim_{r \to \infty} u(r, \theta) \text{ is finite .}
\]

b) Find the solution \( u(r, \theta) \), leaving constants in terms of integrals.

c) What does \( u \) look like far away from the circle (as \( r \to \infty \))? Give an explicit expression (evaluate relevant constants).

---

**P2.** Consider diffusion with a source proportional to the amount (\(\gamma^2 u\)):

\[
\begin{align*}
&u_t = u_{xx} + \gamma^2 u, \quad x \in [0, 2] \\
&u(0, t) = u(2, t) = 0 \\
&u(x, 0) = f(x).
\end{align*}
\]

a) Find all time-independent solutions - show that they only exist for certain values of \(\gamma\).

b) Find the solution. Show that there is only one value of \(\gamma\) such that this convergence to a non-zero steady state is ‘typical’ behavior (i.e. for a ‘random’ \(f(x)\) whose coefficients in the eigenfunction basis are all non-zero).

c) Then find the solution and determine the conditions under which each of the solutions you found in (a) are actually the limit as \(t \to \infty\).
**P3 (a messy problem).** Use eigenfunction expansion to solve

\[ u_{tt} + 2u_t = u_{xx} + e^{-t}x, \quad x \in [0, \pi] \]

\[ u(0, t) = u_x(\pi, t) = 0 \]

\[ u(x, 0) = 0, \quad u_t(x, 0) = 0 \]

(You can use SoV to get \( L \) and the eigenfunctions, only go this far with it). You should leave expansions of functions like \( x \) in terms of integrals; you may also want to define \( \phi_n \) or other constants along the way.

Make sure all constants are well defined. Solve the coefficient IVPs completely and make sure to verify that your solution ‘general’ form is correct for all \( n \).

*Hint: expand \( x \) and \( g(x) \) in terms of the eigenfunctions, defining relevant constants. You’ll need to use undetermined coefficients to solve the ODEs.*

---

**P4.** Use the Rayleigh quotient (multiply by \( \phi \), IBP) to show that

\[-((x^2 + 1)\phi')' + 4\phi = \lambda\phi, \quad \phi'(0) = 4\phi(0), \quad \phi(1) = 0\]

has strictly positive eigenvalues (assuming they exist).

---

**P5 (self-adjoint examples).** In all cases, let \( \langle f, g \rangle \) denote the usual \( L^2 \) inner product on the interval \( (\int_a^b f(x)g(x) \, dx) \).

a) Show that \( L \) with the given BCs in \([0, 1]\) is not self-adjoint:

\[ Lu = u_{xx} + 3u_x, \quad u(0) = u(1) = 0. \]

b) Show that \( L \) with the given BCs in \([0, 2]\) is self-adjoint:

\[ Lu = (r^2u_r)_r + 2u, \quad u_r(2) = 0. \]

c) Show that the following \( L \) is self-adjoint in \([-1, 1]\) even without any BCs imposed:

\[ Lu = ((1 - x^2)u_x)_x. \]