Corrections in red.

Notes:

The problems will be similar to the sort of problems you will see on the exam (roughly 4-5). Note that it is important that you can compute solutions, but also recall key concepts. This set of problems covers most, but not necessarily all, of the material. The review guide contains suggested textbook problems.

Important note: Only one ‘exam-like’ problem on phase planes is included. See textbook, 9.4 (1-12) and 10.1 (1-8) for more. Solutions will be posted to some of these problems.

Solutions will be posted around Monday before the exam.

Problems

R1. a) Show (by direct computation) that the initial value problem
\[ y' = \frac{y}{t} \quad \text{for} \quad t > 0, \quad y(0) = 0 \]
has infinitely many solutions.

b) Show (by direct computation) that the initial value problem
\[ y' = \frac{y}{t} \quad \text{for} \quad t > 0, \quad y(0) = 1 \]
has no solutions.

c) Why does do the results in (a) and (b) not contradict the existence/uniqueness theorem?

R2. Consider the ODE
\[ x^2y'' - 3xy' + 4y = 0. \]

a) One solution is \( y_1 = x^2 \). Use the substitution \( y_2 = x^2v(x) \) to obtain a second solution.

b) Verify that \( y_1 \) and the solution in (a) are linearly independent.

R3 (two true-or-false problems).
For each statement, determine whether it is true or false. If false, briefly explain why not.

I. Consider the system \( \mathbf{x}' = A(t)\mathbf{x} + \mathbf{f} \) in \( \mathbb{R}^n \).
a) The set of solutions to this ODE are spanned by a set of \( n \) solutions.
b) The ODE is equivalent to an \( n \)-th order linear ODE for a function \( y(t) \).
c) Every \( n \)-th order linear ODE is equivalent to a system of this form.
d) If \( A(t) = A \) is a constant matrix, then all solutions exists for all \( t \in \mathbb{R} \).

II. Consider the ODE \( \mathbf{x}' = A \mathbf{x} \) where \( A \) is a \( 2 \times 2 \) real-valued matrix.

a) If \( A \) has two distinct eigenvalues \( \lambda_1 \neq \lambda_2 \) and \( \lambda_1, \lambda_2 \leq 0 \) then all solutions \( \mathbf{x}(t) \) are bounded as \( t \to \infty \).
b) If all the eigenvalues of \( A \) satisfy \( \lambda \leq 0 \) then all solutions \( \mathbf{x}(t) \) are bounded as \( t \to \infty \).
c) A constant solution exists if and only if \( \det(A) = 0 \).

R4. For both parts, \( y \) is a (scalar) function of \( t \).
a) Give an example of an equation \( y' = f(y) \) that has exactly two equilibria and the property that all non-constant solutions are increasing for all \( t \). Sketch the phase line.
b) Give an example of an equation \( y' = f(y) \) such that, for solutions to \( y' = f(y), \ y(0) = y_0 \) it is true that \( \lim_{t \to \infty} y(t) = 2 \) if and only if \( y_0 > -1 \).

R5. Solve the initial value problem
\[
e^x + y^3 \frac{dy}{dx} = 0, \quad y(0) = 1.
\]
What is the interval on which the solution is defined?

R6. a) Find a basis \( \mathbf{x}_1, \mathbf{x}_2 \) for solutions to the system
\[
\mathbf{x}' = A \mathbf{x}, \quad A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}
\]
b) Do the same as in (a), but choose the basis such that
\[
\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
Hint: Use the result in (a).

R7. Consider the ODE
\[
y^{(3)} - 8y = e^t + 3.
\]
a) Write the associated linear system.
b) Find the general solution to the homogeneous problem.
c) Find a particular solution. Hint: don’t use (a) here.
**R8.** Note: again, this is similar to the example from class.

Consider the IVP
\[ x'' + 4x = 1 + \sin \omega t, \quad x(0) = 0, \quad x'(0) = 0. \]

a) Find the solution when \( \omega = 1 \). *Hint: the RHS has two terms; one is easy to deal with.*

b) Find the solution when \( \omega = 2 \).

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**R9.** Consider the IVP
\[ t^3 y' = t^3 - y, \quad y(t_0) = y_0. \]

a) When does the existence/uniqueness theorem guarantee a unique solution?

b) Solve the IVP (you may leave the answer in terms of a definite integral).

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**R10.** Consider the IVP
\[ y' = y^2 - 1, \quad y(0) = 2. \]

a) What does the phase line tell you about the solution?

b) Find the exact solution. *Hint: \( 1/(1 - y^2) = a/(1 - y) + b/(1 + y) \).*

c) What is the interval of existence for the solution to this IVP?

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**R11.** The phase line for
\[ y' = ry - y^3 \]
depends on the parameter \( r \). Identify the ranges of \( r \) where the phase line has a certain qualitative behavior. Draw the phase line in each case; determine the equilibria and their stability. (This is called a pitchfork bifurcation).

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**R12.** Consider the system
\[ \mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{x}. \]

a) Sketch the nullclines and indication the direction (sign of \( x' \) and \( y' \)) in each region between the nullclines.

b) Sketch the phase plane (on a different graph), including any half-line solutions (i.e. solutions along a line in the phase plane). Draw at least one plausible solution in each ‘region’ between these lines.

c) Identify the set of points in the phase plane \( \mathbf{x}_0 \) such that the solution starting at \( \mathbf{x}_0 \) converges to \( (0,0) \) as \( t \to \infty \).
Solutions

R1. a) Formally (ignoring division by $y$ issues), separate and integrate:

$$y' = \frac{y}{t} \implies \frac{1}{y} y' = \frac{1}{t} \implies \log y = \log t + C \implies y = Ct.$$ 

Plug in to check that $y(t) = Ct$ is a solution to the IVP for any $C$.

b) When $y(0) = 1$ part (a) says that all solutions must have the form $y(t) = Ct$ for $t > 0$. But $y(0) = 0$ so $y(0)$ cannot be 1.

c) The ODE is $y' = f(t, y)$ for $f = y/t$. Since $f$ is not continuous at $t = 0$, neither part of the theorem applies.

R2. a) As in the hint, let $y = x^2 v(x)$ and plug in:

$$0 = x^2 y'' - 3xy' + 4y = x^2(2v + 4xv' + x^2v'') - 3x(2xv + x^2v') + 4x^2v = 4x^3v' + x^4v'' - 3x^3v' = x^4v'' + x^3v'$$

since the $x^2 v$ terms cancel. Using an integrating factor,

$$0 = x^3(xv'' + v') = x^3(xv')' \implies v = c_1 + c_2 \log x.$$ 

Only one solution is needed; take $v(x) = \log x$ so $y_2 = x^2 \log x$.

b) Note: technically, $v = \text{const}$ works for (a) but not here. Compute the Wronskian:

$$y_1 y_2' - y_1' y_2 = x^2(x + 2x \log x) - 2x^3 \log x = x^3$$

so the solutions are LI for $x > 0$ (which is where the ODE/solutions are defined anyway).

R3. Note that explanations are included for true as well (not asked for).

I-a) False; the ODE is inhomogeneous (only true when $f = 0$).

I-b) This is false in general. $n$-th order linear ODEs are a special case (e.g. for the standard transformation, $x_1^t = x_2, x_2^t = x_3, \cdots$).

I-c) This is true, by setting $x_1 = y, x_2 = y', \cdots x_n = y^{(n-1)}$ and $x = (x_1, \cdots, x_n)^T$.

I-d) Technically, not true\(^1\) since $f(t)$ was not specified. If $f$ is assumed continuous for all $t$, then true since all the basis solutions have only $t^k e^{\lambda t}$ terms (all exist for all $t \in \mathbb{R}$ and variation of parameters then guarantees the particular solution also exists.

II-a) This is true. The solutions must be $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ (since $\lambda_1, \lambda_2$ have mult. 1) so both exponentials either decay or the solution is constant (also bounded).

II-b) This is false; if $\lambda_1 = \lambda_2 = 0$ solutions can grow like $t$ (multiplicity 2 case).

II-c) This is true; a constant solution requires $\lambda = 0$ to an eigenvalue ($e^M$ is a solution).

\(^1\)Either answer is fine here, as long as it is explained if false.
R4. a) We need $f \geq 0$ for all $y$. Both equilibria must be half stable. One example is $y' = y^2(y - 1)^2$. The phase line is straightforward to draw; see below.

b) We need two equilibria: an unstable one at $-1$ and one stable one at $2$. This is achieved (e.g.) by $y' = -(y + 1)(y - 2)$ (sketched below).

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{phase_line.png}}
\end{array}
\]

R5. This equation is separable:

\[y^3y' = -e^x \implies \frac{1}{4}y^4 = C - e^x \implies y = \pm(C - 4e^x)^{1/4}.
\]

From $y(0) = 1$ we get the IVP solution

\[y = (5 - 4e^x)^{1/4}.
\]

This fails to exist where $y = 0$ (from the ODE $y' = -y^3/e^x$ or the root in the solution):

\[y = 0 \iff x = \log(5/4).
\]

Since this value is $> 0$, the solution starting at $x = 0$ exists in $(-\infty, \log(5/4))$.

R6. Characteristic polynomial, eigenvalues/vectors:

\[p(\lambda) = \lambda^2 + 4\lambda + 4 \implies \lambda = -2, \quad v_1 = (1, -1)^T
\]

(only one eigenvector). This is the repeated root case, so we find a generalized eigenvector:

\[(A + 2I)v_2 = v_1 \implies v_2 = (-1, 0)^T.
\]

The solution is then

\[x(t) = c_1e^{-2t}v_1 + c_2e^{-2t}(v_2 + tv_1).
\]

b) To get the new basis, solve the IVP for the two given ‘initial conditions’ using the old basis above:

\[x_1(0) = (1, 0)^T \implies c_1v_1 + c_2v_2 = (1, 0) \implies c_1 = 0, c_2 = -1
\]

\[x_2(0) = (0, 1)^T \implies c_1v_1 + c_2v_2 = (0, 1) \implies c_1 = -1, c_2 = -1
\]

which gives the solutions

\[x_1 = e^{-2t}\begin{bmatrix} 1-t \\ t \end{bmatrix}, \quad x_2 = e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
R7. Let \( x_1 = y, x_2 = y' \) and \( x_3 = y'' \) and \( \mathbf{x} = (x_1, x_2, x_3) \). Then
\[
\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t + 3 \end{bmatrix}.
\]

b) For the homogeneous problem:
\[
p(\lambda) = \lambda^3 - 8 \implies (\lambda - 2)(\lambda^2 + 2\lambda + 4) \implies \lambda = 2, \ -1 \pm i\sqrt{3}
\]
so the general solution is \( y(t) = c_1 e^{2t} + e^{-t}(c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t) \).

c) Use undetermined coefficients. Split into two parts, \( e^t \) and 1.
For \( e^t \): \( \lambda = 1 \) is not an eigenvalue so guess (let \( Ly = y^{(3)} - 8y \))
\[
y_p = e^t \implies Ly_p = e^t - 8e^t = -7e^t \implies y_p = -\frac{1}{7}e^t.
\]
For \( e^t \): \( \lambda = 1 \) is not an eigenvalue; guess a const. to get \( y = -3/4 \).
In total: \( y_p = -e^t/7 - 3/4 \).

R8. See example from class/notes for the process.
The only differences are \( \lambda = \pm 2i \) (so \( x(t) = c_1 \sin 2t + c_2 \cos 2t \) for the homogeneous part) and there is an additional \(-1/4\) in the particular solution from the 1 term.

R9. a) Since \( y' = 1 - y/t^3 \) and \( f = 1 - y/t^3 \) and \( \frac{2t}{y} = -1/t^3 \) are both continuous except at \( t = 0 \), a unique solution is guaranteed unless \( t_0 = 0 \).
b) This equation is linear. Use an integrating factor \( \phi = \exp(\int 1/t^2) = \exp(-1/2t^2) \) to get
\[
y' + y/t^3 = 1 \implies (e^{-1/(2t^2)}y)' = e^{-1/(2t^2)} \implies y(t) = e^{1/(2t^2)} \left( C + \int_{t_0}^t e^{-1/(2s^2)} \right) ds \)
where (from \( y(t_0) = y_0 \)) the constant is \( C = y_0 e^{-1/(2t_0^2)} \).

R10. a) The equilibria are at \( y = \pm 1 \). From the phase line (draw it!), \( y = 1 \) is unstable, \( y = -1 \) is stable. For \( y(0) = 2, y \to \infty \) as \( t \) increases (see (c)) and \( y \to 1 \) as \( t \to -\infty \).
b) Write
\[
\left( \frac{1}{y-1} - \frac{1}{y+1} \right) y' = 2 \implies \log |y - 1| - \log |y + 1| = 2t + C
\]
then solve for \( y \):
\[
\frac{|y - 1|}{|y + 1|} = Ce^{2t}.
\]

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\(^2\)Value changed from 4 to 8 to fix a typo; the solution was messier but same structure.
Plug in $y(0) = 2$ to get $C = 1/3$. Note that $y > 1$ for all $t$ (from (a)) so the absolute values can be dropped:

$$\frac{y - 1}{y + 1} = \frac{1}{3} e^{2t} \implies y - 1 = \frac{1}{3} e^{2t}(y + 1) \implies y(t) = \frac{1 + e^{2t}/3}{1 - e^{2t}/3}.$$

The solution has $y(t) \to \infty$ where $1 = e^{2t}/3$, so at $t^* = \log(3)/2 > 0$. Since $t_0 = 0$, the interval of existence for this solution is

$$(-\infty, \log(3)/2).$$

This agrees with the phase line (converges as $t \to -\infty$, diverges as $t$ increases).

**R11.** There are three cases (see phase line sketch below):

If $r < 0$ then there is a single equilibrium point at $y = 0$; it is stable.

If $r = 0$ then $y' = -y^3$. Since $y' < 0$ for $y > 0$ and $y' > 0$ for $y < 0$, it is stable.

If $r > 0$ then there are three equilibria, at 0 and $\pm \sqrt{r}$. From the phase line, $\pm \sqrt{r}$ are both stable and 0 is unstable.

**R12.** The relevant information:

$$\lambda_1 = 1, \; v_1 = (1, 2)^T, \quad \lambda_2 = -2, \; v_2 = (1, 1)^T,$$

and the nullclines:

$$x' = 0 \iff y = -3x/2, \quad y' = 0 \iff y = -x$$

This is a saddle with solutions $e^{t}v_1$ leaving $(0, 0)$ and $e^{-2t}v_2$ entering $(0, 0)$. 
1. Selected book problems

See 10.1.1 and 10.1.3 solutions in the back of the book. One example in detail is shown below (to get a sense of what is expected and what is not).

10.1.6. The nullclines are:

\[ \begin{align*}
  x' &= 0 \implies x = 0 \text{ or } y = 1.2 \\
  y' &= 0 \implies y = 0 \text{ or } x = 0.5
\end{align*} \]

Note that \( x' = 0 \implies x = 0, y' = -0.5y \) so \( x = 0 \) is an invariant set and \( y \to 0 \) on the line.\(^3\)

Note that \( y' = 0 \implies y = 0, x' = 1.2x \) so \( y = 0 \) is an invariant set and \( x \to \pm \infty \) on the line.

It follows that each quadrant \((x, y > 0 \text{ etc.})\) is invariant.

There are two equilibria. The Jacobian is

\[ J = \begin{bmatrix}
  1.2 - y & -x \\
  y & x - 0.5
\end{bmatrix} \]

The Jacobian at each equilibrium and eigenvalues and type are

\( (0, 0) \implies J = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.5 \end{bmatrix} \implies \lambda_1 = 1.2, \lambda_2 = -0.5 \implies \text{saddle point} \)

\( (0.5, 1.2) \implies J = \begin{bmatrix} 0 & -0.5 \\ 1.2 & 0 \end{bmatrix} \implies \lambda = \pm i\sqrt{3/5} \implies \text{center} \)

This is enough to sketch a plausible phase portrait. The \( x \) and \( y \) nullclines are in black/green, solutions in purple. Note that with this limited information, we cannot deduce exactly what it should look like (are orbits really closed? Do they diverge off to \( \infty \) or stay bounded?).

The only requirements here are that (i) the directions are correct in each region, (ii) solutions cross the nullclines in the correct way and (iii) solutions near the point at \((0, 0)\) show a saddle point.

\(^3\)Note: on an exam, recognize this fact for sketching; you’d be asked explicitly to supply the argument if it is needed and given a hint if the set is not ‘obvious’ (invariant along a nullcline or trivial solution).