This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Put your initials on the top right of every page, in case the pages become separated. Your signature above indicates your adherence to the honor code:

_I have neither given nor received unauthorized help on this exam, and I have conducted myself within the guidelines of the Duke Community Standard. Moreover, I will not discuss the content of this exam with anyone until authorized to do so._

The following rules apply:

- Put your initials in the upper right of each page, in case they become separated.

- You may **not** use books, notes, or any calculator on this exam. You may freely use given formulas. Use this to avoid unnecessary work.

- If you use a theorem or result from class you must (briefly) explain why it applies. If the theorem is from the formula sheet, also cite it by number.

- Organize your work neatly in the space provided. If you need more space, use the back of the pages, and indicate this clearly.

- **You are required to show your work on each problem.** A correct answer without justification may receive no credit; an incorrect answer with substantially correct work may get partial credit.

- Shorthand you use in your solutions should be well-defined. Symbols etc. not defined in the problem should be defined in the solution.

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1. (15 points) Short answer questions (no justification required).
   (a) (5 points) Find a conserved quantity $E(x, y)$ for the system $x' = xy^2$, $y' = x^3$.

   (b) (4 points) Let $f(x)$ be an $L^2$ function in $[-\pi, \pi]$ such that
   
   $f(x)$ is even and $\int_{0}^{\pi} f(x) \cos nx \, dx = 0$ for $n = 1, 2, 3, \ldots$.

   What can be said about $f$?

   (c) (6 points) For the PDEs listed below, indicate with a circle (one per line) whether it is linear and inhomogeneous (LI), linear and homogeneous (LH) or not linear (N).
   Here $u$ denotes a function $u(x, t)$ and subscripts denote partial derivatives.

   $u_t = x^2 u_{xx} + t$, \hspace{1cm} LI \hspace{1cm} LH \hspace{1cm} N
   $u_t = (u^2)_{xx} + x$, \hspace{1cm} LI \hspace{1cm} LH \hspace{1cm} N
   $u_t = (x^2 u_x)_x$, \hspace{1cm} LI \hspace{1cm} LH \hspace{1cm} N
   $u_{tt} + u = u_{xx}$, \hspace{1cm} LI \hspace{1cm} LH \hspace{1cm} N
2. (25 points) Consider the planar system

\[ x' = y - x, \quad y' = -x + x^2 - y. \]

(a) (10 points) Identify the equilibrium points, the linearization and their type. For saddle points and nodes only, also note the directions (just the vectors) of the ‘half-line solutions’ (stable/unstable manifolds or slow/fast directions) and the corresponding eigenvalues.
(b) (15 points) Draw a plausible phase portrait in a region containing at least the range 
\(-1 \leq x \leq 3, -1 \leq y \leq 3\). Include:

- The nullclines (suggested: draw dashed or thinner to distinguish from solution curves)
- Stable/unstable manifolds of saddles (extend to the whole sketch)
- Solution curves through \((-1, 0)\) and \((0, 2)\) and one other (different) solution.
3. (20 points) Consider the system

\[ \begin{align*}
x' &= -x^3 - y \\
y' &= 3x - y
\end{align*} \]

(a) (12 points) Find a Lyapunov function. *Hint: guess* \( ax^2 + y^2 \) for some \( a \).

(b) (8 points) Determine the set \( S \) of points \((x_0, y_0)\) such that the solution \((x(t), y(t))\) starting at \((x_0, y_0)\) converges to the origin as \( t \to \infty \). Justify your answer.
4. (20 points) Consider the following eigenvalue problem for \( \phi(x) \) in the interval \([0, b]\):

\[-4\phi'' = \lambda \phi, \quad \phi'(0) + a\phi(0) = 0, \quad \phi(b) = 0\]

(a) (8 points) Find the condition(s) on \( a, b \) such that \( \lambda = 0 \) is an eigenvalue and find the corresponding eigenfunction (when it exists).

(b) (12 points) Find the positive eigenvalues and eigenfunctions when \( a = 0 \), i.e. for

\[-4\phi'' = \lambda \phi, \quad \phi'(0) = 0, \quad \phi(b) = 0.\]

You may assume there are no negative eigenvalues.
5. (20 points) Define, in the interval $[0, 1]$, the function

$$f(x) = \begin{cases} 
\frac{1}{2} - x & 0 \leq x \leq \frac{1}{2} \\
0 & \frac{1}{2} \leq x \leq 1
\end{cases}.$$ 

(a) (6 points) Let $f_o$ and $f_e$ be the odd and even periodic extensions with period 2. Sketch these functions in the interval $[-2, 2]$ (note the interval here!).
(b) (14 points) Calculate the Fourier series for $f_o$ (the odd extension). In your answer, indicate which coefficients are zero (if any) and which are non-zero. Simplify as much as possible but you can leave $\sin(n \cdots)$ and $\cos(n \cdots)$ in your answer.

Some integrals:
\[
\begin{align*}
\int (x - a) \sin kx \, dx &= -\frac{1}{k} (x - a) \cos kx + \frac{1}{k^2} \sin kx, \\
\int (x - a) \cos kx \, dx &= \frac{1}{k} (x - a) \sin kx + \frac{1}{k^2} \cos kx
\end{align*}
\]