MIDTERM I REVIEW PROBLEMS

MIDTERM DATE: OCT. 2

NOTES:

The problems here are a mix of computational and conceptual problems. They will be similar to the sort of problems you will see on the exam. Note that it is important that you can compute solutions, but also explain key concepts. This set of problems does not cover all the material.

Solutions can be found after the questions.

PROBLEMS

R1. a) Explain why the following argument, which shows that inhomogeneous and homogeneous linear ODEs are equivalent, is wrong. Consider a linear, inhomogenous ODE for \( y(t) \),

\[
L[y] = f
\]

where \( L[y] = y' + p(t)y \) and \( f \) is a function of \( t \). Define

\[
M[y] = L[y] - f.
\]

Then (1) is equivalent to the linear homogeneous ODE

\[
M[y] = 0.
\]

b) What is the relation between solutions to the homogeneous ODE \( L[y] = 0 \) and inhomogeneous ODE \( L[y] = f \)?

R2. a) Show that the initial value problem

\[
y' = y/t \text{ for } t > 0, \quad y(0) = 0
\]

has infinitely many solutions.

b) Show that the initial value problem

\[
y' = y/t \text{ for } t > 0, \quad y(0) = 1
\]

has no solutions.

c) Why does do the results in (a) and (b) not contradict the existence/uniqueness theorem?
R3. For each statement, determine whether it is true or false. If false, briefly explain why not. (Useful exercise: if true, also explain why).

I. Consider the system $x' = A(t)x + f$ in $\mathbb{R}^n$.

   a) The set of solutions to this ODE are spanned by a set of $n$ solutions.
   
   b) The ODE is equivalent to an $n$-th order linear ODE for a function $y(t)$.

II. Consider the ODE $x'(t) = Ax(t)$ where $A$ is an $n \times n$ matrix.

   a) The ODE has a constant solution (i.e. independent of $t$) if and only if $\det(A) = 0$.
   
   b) The function $x(t) = (\cos t)v$ can be a solution (where $v$ is a vector).
   
   c) If $x(t)$ is a solution then $x(t + c)$ is a solution for any $c \in \mathbb{R}$.

R4. Consider the ODE

$$t^2y'' - 3ty' + 4y = 0.$$ 

a) Find all solutions of the form $y = t^r$ by direct substitution (there should only be one).

b) To get a second solution, let

$$y_2 = v(t)y_1.$$ 

Substitute this into the ODE and then solve for $v$ to get a second solution linearly independent to $y_1$. *Hint: you should get an ODE for $v'$ that you can solve.*

R5. Consider the ODE

$$y' = y^\alpha, \quad y(0) = 1$$

for $y(x)$ where $\alpha > 0$.

a) Show that if $\alpha < 1$ then the solution exists for all $x \in \mathbb{R}$, but if $\alpha > 1$ then the solution blows up ($|y| \to \infty$) at some critical value of $x$ (which you should calculate).

b) Is the same true if the initial condition is negative?

R6. For both parts, $y$ is a (scalar) function of $t$.

a) Give an example of an equation $y' = f(y)$ that has exactly two equilibria and the property that all non-constant solutions are increasing for all $t$.

b) Give an example of an equation $y' = f(y)$ such that, for solutions to

$$y' = f(y), \quad y(0) = y_0$$

it is true that $\lim_{t \to \infty} y(t) = 2$ if and only if $y_0 > -1$. 
**R7.** a) Solve the initial value problem

\[
(x + y) + (x + 2y) \frac{dy}{dx} = 0, \quad y(2) = 3.
\]

b) What is the interval on which the solution is defined?

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**R8.** Identify whether the statement is true or false. In either case, briefly explain why it is true/false.

Consider the ODE

\[
ay'' + by' + cy = 0
\]

where \(a, b, c\) are real numbers and \(a \neq 0\).

a) If \(y(t)\) is a solution then \(y'(t)\) is also a solution.

b) If \(y(t)\) is a solution then \(y\) and \(y'\) are a basis for solutions.

c) If \(y(t)\) is a solution then \(y\) and \(y'\) are always linearly dependent.

d) A solution \(y(t)\) can have exactly one local maximum.

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**R9.** Find a basis \(x_1, x_2\) for solutions to the system

\[
x' = Ax, \quad A = \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}
\]

such that

\[
x_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

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**R10.** Consider the ODE

\[
y^{(3)} - y = t^2.
\]

a) Write the associated linear system.

b) Find a basis for solutions to the homogeneous problem and a particular solution. *Hint: don’t use (a) here.*
R1. The argument in (a) is wrong since $M$ is not a linear operator due to the $f$ term:

$$M[c_1y_1 + c_2y_2] = L[c_1y_1 + C_2y_2] - f = c_1L[y_1] + c_2[y_2] - f$$

but

$$c_1M[y_1] + c_2M[y_2] = c_1L[y_1] + c_2L[y_2] - (c_1 + c_2)f.$$ 

The two are not equal for all $c_1, c_2$ (only true if $c_1 + c_2 = 1$).

b) The general solution to $L[y] = f$ is the sum of a homogeneous and a particular solution.

R2. See lecture notes on existence/uniqueness (this was an example from class).

R3.

Part I:

a) False; only true if the equation is homogeneous.

b) False; the statement is true in reverse.

Part II:

a) True; $x' = 0$ for all $t$ means $Ax = 0$, so $x$ must be a constant vector in the null space of $A$.

b) False; $x(\pi/2) = 0$ which would imply that $x(t) = 0$ for all $t$ (by the uniqueness theorem).

c) True; the equation is autonomous (or note that $(x(t+c))' = x'(t+c) = A(x(t+c))$).

R4. Plug in $t^r$:

$$r(r-1)t^r - 3rt^r + 4t^r = 0.$$ 

Factor out the $t^r$ to get

$$(r - 2)^2 t^r = 0.$$ 

Thus $t^r$ is a solution only for $r = 2$. Let $y_1 = t^2$ and

$$y_2 = vy_1.$$ 

Plugging into the ODE, we get

$$t^2(t^2v'' + 2(2t)v' + 2v) - 3t(t^2v' + 2tv) + 4t^2v = 0.$$ 

The terms with $v$ cancel, leaving

$$t^4v'' + t^3v' = 0.$$ 

This is an ODE for $v'$:

$$v'' + \frac{1}{t}v' = 0.$$ 

Multiplying by the integrating factor $t$:

$$(tv')' = 0 \implies v' = a/t.$$
Thus

\[ v = a \ln t + b \implies y_2 = at^2 \ln t + bt^2. \]

Since we are seeking a second basis solution, take \( b = 0 \) and \( a = 1 \). The solution is

\[ y_2 = t^2 \ln t. \]

The \( \ln t \) here is the analogue of the factor \( t \) needed for multiple roots with constant coefficient equations (to be revisited in our study of series solutions in Chapter 5).

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**R5.** This equation is separable. If \( \alpha \neq 1 \) then

\[ y^{-\alpha}y' = 1 \implies \frac{1}{1 - \alpha}y^{-\alpha + 1} = t + C. \]

Let \( \beta = 1 - \alpha \). Then

\[ y^\beta = \beta t + C. \]

Applying the initial condition \( y(0) = y_0 \) we get \( C = 1 \) so

\[ y = (\beta t + 1)^{1/\beta}. \]

If \( \beta > 0 \) then clearly this solution exists for all \( t > 0 \). However, if \( \beta < 0 \) then the solution is infinite when \( t \) reaches the point

\[ t^* = -1/\beta = \frac{1}{\alpha - 1}. \]

If \( \alpha = 1 \) then it is easy to compute the solution \( y = e^t \), which also exists for all \( t > 0 \).

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**R6.**

a) We need \( f \geq 0 \) for all \( y \). Both equilibria must be half stable. One example is \( y' = y^2(y - 1)^2 \).

b) We need two equilibria: an unstable one at \(-1\) and one stable one at \(2\). This is achieved (e.g.) by \( y' = -(y + 1)(y - 2) \) (phase line should be drawn here).

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**R7.** This equation is exact. The ‘potential’ \( \phi(x, y) \) satisfies

\[ \frac{\partial \phi}{\partial x} = x + y, \quad \frac{\partial \phi}{\partial y} = x + 2y. \]

Integrating the first equation with respect to \( x \), we find

\[ \phi = \frac{1}{2}x^2 + xy + h(y), \]

and differentiating with respect to \( y \) we find that \( h(y) = y^2 \), so the solution is

\[ \frac{1}{2}x^2 + xy + y^2 = C. \]

Applying the initial condition, the solution \( y(x) \) is defined implicitly by

\[ \frac{1}{2}x^2 + xy + y^2 = 17. \]
It fails to exist where \( y' \to \infty \), which occurs where \( x = -2y \) (according to the ODE). For the solution \( y(x) \), we will have \( y' \to \infty \) at the points
\[
\frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2 = 17,
\]
or \( x = \pm 2\sqrt{17} \). The solution is therefore defined on \((-2\sqrt{17}, 2\sqrt{17})\).

**R8.**

a) True; differentiate the ODE

b) False; consider \( y = e^{\lambda t} \) (then \( y' \) is a multiple of \( y \)).

c) False; consider \( y = \sin t \) and \( y' = \cos t \).

d) True; \( y = te^{-t} \), for instance.

**R9.** Eigenvalues/vectors:

\[
\lambda_1 = 3, \ v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = -1, \ v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

One way to do this is to have the basis functions solve the IVPs

\[
x' = Ax, \quad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

and

\[
x' = Ax, \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

for \( x_1, x_2 \) respectively. Ignoring this for now, the general solution is:

\[
x = c_1e^{3t}v_1 + c_2e^{-t}v_2.
\]

First solve the IVP with \( x(0) = (1, 0) \):

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1v_1 + c_2v_2 \implies c_1 = -1, \quad c_2 = -1.
\]

Now solve the same for \( (0, 1) \):

\[
\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1v_1 + c_2v_2 \implies c_1 = -2, \quad c_2 = -1.
\]

The desired basis is then

\[
x_1 = -e^{3t}v_1 - e^{-t}v_2, \quad x_2 = -2e^{3t}v_1 - e^{-t}v_2
\]

**R10.** Let \( x_1 = y, x_2 = y' \) and \( x_3 = y'' \). Then the system is

\[
x_1' = x_2, \quad x_2' = x_3, \quad x_3' = t^2 + x_1.
\]

In matrix form, this is
\[
x' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ t^2 \end{bmatrix}
\]
b) The characteristic polynomial is \( r^3 - 1 = (r - 1)(r^2 + r + 1) \) with roots
\[
\lambda = 1, \quad \lambda = \frac{1}{2}(-1 \pm \sqrt{-3}) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.
\]
The (general) homogeneous solution is then
\[
y_h = c_1 e^t + c_2 e^{-t/2} \cos \frac{\sqrt{3}}{2} t + c_3 e^{-t/2} \cos \frac{\sqrt{3}}{2} t.
\]
A particular solution is (by guessing \( at^2 \))
\[
y_p = -t^2.
\]
The general solution is \( y = y_h + y_p \).