The second midterm will take place on **Thursday Nov. 15** in class. No new topics will be added beyond this list. The midterm covers material through the **Thursday before the exam**. No topics will be added; the list will be updated depending on Thursday’s class.

**Formulas provided**

The exam is **closed book and closed notes**. You will be given the following formulas (without context) if they are relevant to the exam problems:

- Trig identities for double/half angles and tedious integral formulas
- The statements of the two Lyapunov theorems (without defining terms like positive definite)
- The statements of the theorem on linearization and stability for non-linear centers
- The Fourier series formula (form of the series and coefficients). Note that you may need to do some variation where the formula does not apply directly.
- Statement of the theorem that says eigenfunctions form an orthogonal basis

**Some advice**

- **Review background** (see next section). We need to solve (simple) ODEs to solve PDEs!

- **Understand what you need to solve the problem.** We have some powerful general methods, but they take work. A complete solution is often not required to answer a question. For instance, if asked to show that all solutions

  \[ u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \phi_n(x) \]

  to a certain heat equation problem converge to zero, it is enough to show that \( \lambda_n > 0 \) for all \( n \); finding \( b_n \) in terms of the initial condition is unnecessary.

- **Use computational shortcuts.** Use the tricks for integrating odd/even functions. If you know only a few modes will be present in a solution, write \( u(x, t) \) only in terms of those eigenfunctions instead of solving for all the (mostly zero) terms.

- **Be very careful with domains.** For convergence, the periodicity of the function can matter (if its endpoint values do not match). The formulas for the Fourier series, inner products and so on all depend on the interval.
Key background

It is important that you are comfortable with the practical calculations we learned for ODEs from the first half of the course and some miscellaneous tricks. This includes:

- **Integrating factors** first order linear ODEs. Such equations arise when solving for the coefficients $a_n(t)$ in an eigenfunction series, e.g. equations like
  \[
a_n' + \lambda_n a_n = t
  \]
- **Second order, constant coefficient ODEs** with a parameter. Given an equation like
  \[
y'' + y' + ky = 0
  \]
you should be able to solve it (find the general solution) and identify the cases (complex roots, repeated, real and distinct) depending on $k$ (see: eigenvalue problems).
- **Linear systems of ODEs**: Used for linearization.
- **Theory for linear ODEs**: (linear independence, homogeneous/particular solutions, basis etc.). These concepts show up in various places (Fourier series; eigenvalue problems; systems of ODEs).

Topics

All topics are covered in lecture notes. Section numbers refer to the book. The notation will match that used in class and the lecture notes.

Topics marked in red will not be on the midterm.

- Nonlinear ODEs
  - Notable omissions:
    - limit cycles (e.g. the Van der pol homework problem)
  - Concepts:
    - Asymptotic stability; Lyapunov stability
    - Stable/unstable manifolds (for saddle points)
    - Uniqueness/non-intersection
  - Phase planes, calculations:
    - Linearization at an equilibrium point
    - Accurate phase planes for linear constant-coefficient planar systems
    - Precise drawings for spirals/centers
    - Phase planes for non-linear systems (equilibria; nullclines; linearization etc.)
  - More on non-linear systems
    - Lyapunov functions, conserved quantities (showing $\dot{V} \leq 0$ or $\dot{E} = 0$)
– Proving stability using Lyapunov’s theorems
– Using the conserved quantity to draw the phase portrait
– Invariant sets; using this and uniqueness to show solutions stay in some region (e.g. the first quadrant in examples from homework)
– informal arguments to deduce paths of solutions (using the direction field, direction along nullclines)
– gradient systems

• PDEs
  ○ Notable omissions:
    – Non-homogeneous PDEs (all PDEs on the midterm will be homogeneous)
  ○ Calculations (PDEs)
    – Solve eigenvalue problems (including showing no solutions in various cases)
    – Solve the (homogeneous) heat equation in a bounded domain with nice boundary conditions using eigenfunctions
    – Determine coefficients using orthogonality of eigenfunctions
    – Detailed analysis of the eigenvalues in nasty cases ($\mu = \tan \mu$ etc.)
  ○ Concepts (PDEs)
    – Superposition and linearity
    – What it means to be an orthogonal basis for $L^2$, the inner product
    – Behavior as $t \to \infty$ (using the eigenvalues)
    – Sets of eigenfunctions form an orthogonal basis for functions that satisfy the boundary conditions

Suggested problems

• Review problems (to be provided separately) and textbook problems (see below). Note that some of them (especially those that ask for a complete phase portrait) are more work than you would be required to do on an exam; a subset of the problem might appear.

Nonlinear systems (fundamentals, phase planes, linearization):

• Section 9.3: 16-23
• Section 9.4: 1-12
• Section 10.2: 1-8 (you may also want to try sketching the phase plane), 20
• Section 10.3: 1-4, 5-8, 13-16 (some were on HW)

Nonlinear systems (conservative systems, Lyapunov functions):
• Section 10.5: 1-10. For 1 and 2, you can also draw the phase plane (by hand) using the level sets. The others are harder to do by hand. You can also identify the equilibria and their type (saddle, center, etc.) and try sketching the phase plane.

• Section 10.5: 21-24 (problems with changing energy)

• Section 10.6: 1, 5, 21

• Section 10.7: 13-26.

PDEs:

• Section 13.2: 5-17

• (more problems to be added since the book does not offer much)