Notes: Corrected a typo in P3. Read problems P3-P5 carefully. Note that for the most part, you are given the solution obtained via the eigenfunction method; the focus here is on concepts over computation.

Problems:

P1 (some eigenvalue problems). Solve the following eigenvalue problems:

a) \(-\phi'' = \lambda \phi, \quad \phi'(0) = 0, \quad \phi(1) = 0\)

b) \(-\phi'' = \lambda \phi, \quad \phi(0) = 0, \quad \phi'(L) = 0\)

c) \(-\phi'' - 2\phi' = \lambda \phi, \quad \phi(0) = 0, \quad \phi(2) = 0\).

In each case, find the eigenvalues and eigenfunctions. Be sure to check all the cases (to verify that the eigenvalue/functions you find are the only ones).

d) Consider the eigenvalue problem for \(L = -d^2/dx^2\) with periodic boundary conditions,

\[-\phi'' = \lambda \phi, \quad \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi)\].

What are the eigenvalues and the corresponding eigenfunctions? Hint: first show there are none for \(\lambda < 0\); the \(\lambda > 0\) case will not just produce one eigenfunction per eigenvalue!

e) For each eigenvalue \(\lambda\) from (d), define the space

\[V_\lambda = \{\phi \text{ solves (1) for the eigenvalue } \lambda\}\].

Verify that this is a vector space. For each eigenvalue, find the dimension and a basis (you should use the results in (d) here).

P2 (odd and even functions).

a) Show that if \(g(x)\) is an \textbf{even function} \((g(x) = g(-x)\) for all \(x)\) then

\[\int_{-L}^{L} g(x) \, dx = 2 \int_{0}^{L} g(x) \, dx.\]

b) Show that if \(g(x)\) is an \textbf{odd function} \((g(x) = -g(-x)\) for all \(x)\) then

\[\int_{-L}^{L} g(x) \, dx = 0.\]

c) Use (b) to efficiently compute

\[\int_{-1}^{1} x^4 + x \cos x + x^4 \sin x \, dx.\]
d) Show that $\cos(mx)$ and $\sin(nx)$ are orthogonal on $[-\pi, \pi]$ for any integers $m$ and $n$.

**P3 (scaling).** Consider the heat equation with Dirichlet BCs in the domain $[0, L]$,

\begin{equation}
\begin{aligned}
&u_t = ku_{xx}, \quad x \in (0, L), \ t > 0 \\
&u(0, t) = u(L, t) = 0, \quad t > 0
\end{aligned}
\end{equation}

where $k > 0$. The constant $k$ is the 'diffusion' coefficient or 'diffusivity'; it has units of area per time and measures the rate at which $u$ 'spreads out'.

The goal here is to show that by a nice change of variables, this PDE is equivalent to the simpler problem with $k = 1$ and $L = \pi$.

a) Defined scaled variables

$$\tau = t/T, \quad \xi = \pi x/L.$$ 

Show that for the right choice of the constant $T$, (2) becomes the following problem for $u(\xi, \tau)$:

\begin{equation}
\begin{aligned}
&u_{\tau} = u_{\xi\xi}, \quad \xi \in (0, \pi), \ \tau > 0 \\
&u(0, \tau) = u(\pi, \tau) = 0, \quad t > 0
\end{aligned}
\end{equation}

b) Suppose we are given that the (general) solution to (3) is

$$u(\xi, \tau) = \sum_{n=1}^{\infty} a_n e^{-n^2\tau^2} \sin n\xi.$$ 

Use part (a) to find the solution $u(x, t)$ to (2) (don’t solve (2) directly).

c) Suppose the initial condition for (2) is

$$u(x, 0) = f(x).$$

Find the corresponding initial condition for (3) (i.e. $u(\xi, 0) = \cdots$).

d) We also know that if the initial condition to (3) is

$$u(\xi, 0) = g(\xi)$$

then the coefficients $a_n$ for the solution in (b) are given by

$$a_n = \frac{2}{\pi} \int_0^\pi g(\xi) \sin(n\xi) \, d\xi.$$ 

Use (c) and this formula to verify that

$$a_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) \, dx,$$

which is the formula you would get if you solved (2) directly.
P4 (superposition). Suppose $u$ and $v$ solve the IBVPs
\begin{align*}
    u_t &= u_{xx} + x^2, \quad x \in (0, 1), \ t > 0 \\
    u(0, t) &= u(1, t) = 0, \quad t > 0 \\
    u(x, 0) &= f_1(x)
\end{align*}
and
\begin{align*}
    v_t &= v_{xx} - xt, \quad x \in (0, 1), \ t > 0 \\
    v(0, t) &= v(1, t) = 0, \quad t > 0 \\
    v(x, 0) &= f_2(x)
\end{align*}
Determine the IBVP satisfied by $w = u + v$.

P5 (Modes and dominant terms). For this problem, assume we know that the solution to the IBVP
\begin{align*}
    u_t &= u_{xx}, \quad x \in (0, \pi), \ t > 0 \\
    u(0, t) &= u(\pi, t) = 0, \quad t > 0 \\
    u(x, 0) &= f(x)
\end{align*}
is
\[ u(x, t) = \sum_{n=1}^{\infty} f_n e^{-n^2 t} \sin nx \]
where $f$ has the eigenfunction expansion
\[ f = \sum_{n=1}^{\infty} f_n \sin nx \]
for coefficients $\{f_n\}$.

a) Let $u_n(x, t) = f_n e^{-\lambda_n t} \sin nx$ (the $n$-th ‘mode’ of the solution). Show (by direct substitution) that $u_n$ solves (6) but with the initial condition
\[ u(x, 0) = f_n \sin nx. \]
This is the important principle that the $n$-th mode of the solution depends only on the $n$-th mode of the initial condition (each mode evolves independently of the others).

b) Use (a) to (quickly) find the solution to the IBVP
\begin{align*}
    u_t &= u_{xx}, \quad x \in (0, \pi), \ t > 0 \\
    u(0, t) &= u(\pi, t) = 0, \quad t > 0 \\
    u(x, 0) &= 4 \sin 2x + \sin 5x.
\end{align*}
c) As $t \to \infty$, the solution $u(x, t)$ decays like $u \sim C e^{-\alpha t}$. What is the rate of decay $\alpha$? 
**Hint:** ignore small terms. Describe how to identify this at a glance from (7).

d) Generalize (d) to the more general IBVP of part (b) (you can assume an infinite sum of small terms is still small here). Does the rate of decay depend on the initial condition $f(x)$, and if so, in what way?