Problems:

• Section 10.7: 4, 17, 21. (For 21, just show that the origin is stable; showing asymptotic stability is optional).

Additional problems:

P1 (epidemics). The Kermack-McKendrick model describes the outbreak of a disease\(^1\). Let \(x(t)\) and \(y(t)\) be the population of healthy and sick individuals. The equations are

\[
\begin{align*}
\dot{x} &= -kxy, \\
\dot{y} &= kxy - ay, \\
\end{align*}
\]

with \(a, k > 0\).

a) Draw the nullclines/vector field/fixed points in the first quadrant. Identify any useful invariant sets. Determine the type of the fixed points (from the linearization). Can you deduced the complete phase portrait from this information?

b) Find a conserved quantity using the \(dy/dx\) trick. Use it to sketch a complete phase portrait and determine what happens to solutions as \(t \to \infty\).

c) Suppose there is a population of \(x_0\) healthy individuals with a small number \(y_0\) of sick individuals \((0 < y_0 \ll x_0)\). What happens? Describe in terms of the model but justify your answer using (b). The answer may depend on \(x_0\).

P2 (gradient systems). A gradient system is an ODE where the vector field comes from the gradient of a function \(\phi(x)\) (called a potential). Such systems are common in physics.

Consider a gradient system in the plane, with potential \(\phi(x, y)\), given by

\[
\begin{align*}
\dot{x} &= -\frac{\partial \phi}{\partial x}, \\
\dot{y} &= -\frac{\partial \phi}{\partial y}. \\
\end{align*}
\]

a) Show that \(\dot{\phi} \leq 0\) and that \(\dot{\phi} = 0\) only at a critical point of \(\phi\) (i.e. at a point where \(\nabla \phi = 0\)).

b) Suppose all the critical points of \(\phi\) are isolated. What types of equilibria (saddles, nodes etc.) are not possible for the system? Justify your answer.

c) Show that periodic orbits cannot exist for this system. Hint: suppose \(x(t)\) is a periodic orbit with period \(T\); compare the values of \(\phi\) at \(x(t)\) and \(x(t + T) = x(t)\) using (a).

\(^1\)Problem adapted from Strogatz’ *Nonlinear Dynamics and Chaos*
**P3 (pendulum).** The angular position \( \theta(t) \) of a simple pendulum is governed by the equation

\[
\dot{\theta} = v, \quad \dot{v} = -\sin(\theta)
\]

where \( v \) is the angular velocity. Note that because \( \theta \) is an angle, the 'phase plane' in \([0, 2\pi]\) should look the same in \([2\pi, 4\pi]\) and so on.

a) Show that

\[
E(\theta, v) = \frac{1}{2}v^2 - \cos \theta
\]

is conserved.

b) Show that \((0, 0)\) is a center (you’ll need to appeal to the theorem here).

c) Show that \((\pm \pi, 0)\) is a saddle point. Give a physical interpretation - why does it make sense this is unstable? *Hint: what should happen if the pendulum starts at \( \theta = \pi \)?*

d) Sketch the phase portrait in \([-2\pi, 2\pi]\). Note that there is some repetition here, but it is useful to draw more than one period to visualize the phase plane better. *Hint: figure out where the saddle point unstable/stable manifolds go first.*
SECOND PART

**P4: Eigenvectors in \( \mathbb{R}^n \).** In this problem, you will work through the (familiar) details of a particular approach to solving linear systems. The point here is to make an analogy to the way we will solve PDEs, so keep this example in mind as we progress through the subject.

Let \( A \) be an \( n \times n \) invertible, symmetric, real-valued matrix. We know this matrix has eigenvectors \( v_1, \ldots, v_n \) and distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \) and that the \( v_i \)'s form a basis. Moreover, the basis is **orthogonal**, i.e.

\[
    v_i \cdot v_j = 0 \quad \text{for} \quad i \neq j.
\]

Suppose now that we wish to solve

\[
    Ax = b.
\]

a) Explain why we can assume that \( x \) and \( b \) have the form

\[
    x = c_1 v_1 + \cdots + c_n v_n, \quad b = b_1 v_1 + \cdots + b_n v_n.
\]

b) By taking the dot product with \( v_i \), show that the coefficients \( b_i \) are given by the formula

\[
    b_i = \frac{b \cdot v_i}{v_i \cdot v_i}.
\]

c) Show that the coefficients \( c_i \) are given by

\[
    c_i = \frac{b_i}{\lambda_i}.
\]

**P5: Some integrals from class.** Let \( m \) and \( n \) be positive integers.

a) Show that when \( m \neq n \),

\[
    \int_0^\pi \sin(mx) \sin(nx) \, dx = 0.
\]

It will be useful to recall the addition identity

\[
    \sin(a) \sin(b) = \frac{1}{2} (\cos(a - b) - \cos(a + b)).
\]

*Hint: note that \( \sin(k\pi) = 0 \) if \( k \) is an integer.*

b) If \( n \) is a positive integer, show that

\[
    \int_0^\pi \sin^2(nx) \, dx = \frac{\pi}{2}.
\]