Notes: Use colors to distinguish solution curves, nullclines, manifolds and so on. You may want to draw detailed nullclines/vector arrows on one plot and then put less detail on a final plot with solutions.

You do not need to be completely rigorous with arguments here. Some features on the phase plane may be impossible to prove for sure without tools we are not covering here.

Larger drawings are easier to work with. Be careful with nullclines that are close together! For instance, an equilibrium point might have two nullclines through it and a stable/unstable manifold, all with different slopes. A perfect diagram is challenging to draw; try to make it as clear as possible.

Explaining some features in text (vs. on the plot) can help minimize clutter as well.

For computer generated plots, use pplane or some other software.

Problems:

- Section 10.3: 11, 16, 17(ac). If you want, take \( a = b = 1/2 \). Part (b) is optional.

Additional problems:

P1 (invariant sets). A set \( S \) in the phase plane is called an invariant set for a system if solutions that start in \( S \) at \( t = 0 \) stay in \( S \) for all \( t \).

The set is called positively invariant if it is true for \( t \geq 0 \) and negatively invariant if it is true for \( t \leq 0 \).

Note: often, ‘invariant’ is used to mean ’positively invariant’ since we usually care about moving ’forward’ in time. The book does this.

a) Explain why every solution curve is an invariant set.

b) Let \( C > 0 \). Show that the set

\[ \{ x^2 + y^2 < C \} \]

is a positively invariant set for

\[ x' = -x, \quad y' = -y. \]
c) Consider again the system
\[ x' = x(-y + 1 - x), \quad y' = y(x + 1 - y). \]
Explain (carefully) why the first quadrant is an invariant set. Then show that the box
\[ \{(x, y) : 0 < x < 1, \quad 0 < y < 2\} \]
is a positively invariant set. *Hint: find the vector field on the boundaries of the box and show that they point inward.*

**P2 (degenerate node).** In class we looked at

\[ x' = x(-y + 1 - x), \quad y' = y(x + 1 - y). \] (1)

Here you fill in some details and explore the system a bit more.

a) Linearize the equation around the equilibrium at \[ C = (0,1) \] (set \[ u = x \] and \[ v = y - 1 \] and find the linearized system for \((u, v)\)). Solve the system explicitly and plot the phase plane.

b) Describe what happens for the actual system (1) near \( C \).

c) Do the same as in (a), but instead of linearizing, keep the \( u^2 \) term in the \( u' \) equation. Solve the equation explicitly (you should get independent equations for \( u' \) and \( v' \)) and sketch the phase plane. Does this match the behavior you described in (b)?

**P3 (more things can happen).** Consider the *Van der Pol oscillator*
\[ x' = y, \quad y' = -x + y(1 - x^2). \]
a) Identify the equilibria and the type/stability predicted by the linearization. Draw the nullclines/directions and take a guess at what the phase portrait should look like.

b) Plot the phase plane on a computer. Is the linearization from (a) accurate? Describe any interesting features you see. What happens to solutions as \( t \to \infty \)?