Note: The last two problems are partly review; the results there will be used directly in next week’s classes so they are important.

Problems:

- 2.9.21 (you don’t have to sketch solutions in every region; just a few), 2.9.29
- Read and review Chapter 7.
- Suggested problems (not graded and you don’t have to turn them in, but make sure you know how to answer them): 7.2.9; solve the systems in 7.3.1 and 7.3.3 (by whatever method), 7.5.19, 7.6.19, 7.7.53 and 7.7.54.

Additional problems:

P1 (more bifurcations).

a) Consider the ODE

\[ y' = ry - y^3 \]

where \( r \) is a parameter. As in the fish example from class, identify the possible (qualitative) phase lines and then draw a bifurcation diagram (i.e. plot the equilibria as a function of \( r \)). Describe what happens.

Note: This is called a ‘pitchfork bifurcation’.

b) Do the same for the ODE

\[ y' = ry - y^2. \]

Note: This is a ‘transcritical bifurcation’.

P2 (behavior allowed on a phase line).

a) Can an autonomous first order ODE \( y' = f(y) \) have two stable equilibrium points and no other equilibrium points? What about two half-stable equilibrium points and no others?

b) Can \( y' = f(y) \) have periodic solutions (other than constant solutions)? Note: assume the uniqueness theorem holds and solutions are continuous. A function \( y(t) \) is periodic with period \( T \) if it holds that

\[ y(t) = y(t + T) \quad \text{for all } t. \]

For instance, \( \sin t \) and \( \cos t \) are periodic but \( t \sin t \) is not.

c) Given a phase line for \( y' = f(y) \), explain how to draw a phase line that shows the behavior of solutions as \( t \to -\infty \). Do so for \( y' = y^2 + y^3 \).
d) A system is said to be reversible (in time) if solutions forward in time are solutions backward in time and vice versa. In terms of ODEs,

\[ y(t) \text{ is a solution} \iff y(-t) \text{ is a solution.} \]

Physically, this means that after running the system for some time, that state can be undone.\(^1\)

Show that \( y'' + y = 0 \), which models an oscillating object, is reversible. Is it possible for \( y' = f(y) \) to be reversible?

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**P3 (Linearity).**

a) Which of the following operators are linear? For one of the linear examples, prove that is linear from the definition. If it is non-linear, briefly explain why (what terms are a problem?).

i) \( L[y] = y' + p(t)y - f(t) \) where \( p, f \) are continuous functions

ii) \( L[y] = y' + p(t)y \)

iii) \( L[y] = y'' \)

iv) \( L[y] = \int_0^t f(s)y(s) \, ds \) where \( f \) is a continuous function.

v) \( L[y] = y' - y^2 \)

b) Consider the linear ODE

\[ L[y] = 0 \]

where \( L \) is a linear operator. Show that the set of solutions

\[ \{ y : L[y] = 0 \} \]

is a vector space. *Note: you may assume multiplication, addition etc. are all well-defined.*

c) Consider the first-order ODE

\[ y' + p(t)y = 0. \]

Let \( L[y] = y' + p(t)y \). Determine the dimension (i.e. the size of a basis) for the set

\[ V = \{ y : L[y] = 0 \}. \]

*Hint: you have the exact general solution already.*

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**P4 (some linear algebra review).**

a) Suppose an \( n \times n \) matrix \( A \) has linearly independent eigenvectors \( v_1, v_2, \ldots, v_n \) with \( Av_k = \lambda_k v_k \) for \( k = 1, \ldots, n \). Let \( V \) be the matrix whose columns are the eigenvectors. Show that

\[ A = VDV^{-1} \]

where \( D \) is the diagonal matrix of eigenvalues. *Hint: multiply by \( V \).*

b) What are some conditions that guarantee that the eigenvectors of an \( n \times n \) matrix \( A \) form a basis for \( \mathbb{R}^n \)?

\(^1\)For instance, the motion of a frictionless pendulum is reversible; if there is friction it is not. One consequence is that for a video of the process, you cannot tell whether it is being played forwards or backwards (both look valid). See [https://www.youtube.com/watch?v=p08_K1TKP50](https://www.youtube.com/watch?v=p08_K1TKP50) for a much more striking example of reversibility.
c) What is the difference between the algebraic and geometric multiplicities of an eigenvalue?

d) Consider the matrix
\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \]
Find all the eigenvalues/eigenvectors of \( A \). Is there a basis of eigenvectors of \( A \) for \( \mathbb{R}^3 \)?

e) Let \( v_1 \) be an eigenvector of \( A \). Find vectors \( v_2 \) and \( v_3 \) such that
\[ v_1 = (A - I)v_2, \quad v_2 = (A - I)v_3. \]
Show that \( v_1, v_2, v_3 \) form a basis for \( \mathbb{R}^3 \). Note: These are 'generalized eigenvectors'.

f) (optional; we'll cover this briefly in class) How does one construct a basis of generalized eigenvectors of a matrix \( A \) when there is no basis of eigenvectors?