Math 353 Lecture Notes
Orthogonal bases, the heat equation

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Topics covered

- Linear algebra (review in $\mathbb{R}^n$)
- Deriving the heat equation
- A first PDE example: the heat equation
- Function spaces: introduction to L^2

1 Linear algebra: orthogonal bases in $\mathbb{R}^n$

Here a review of linear algebra introduces the framework that will be used to solve differential equations. The structure we review for **vectors** and **matrices** in the space $\mathbb{R}^n$ to solve linear systems $Ax = b$ will be adapted to **functions** and **linear operators**.

The familiar setting for linear algebra is the space $\mathbb{R}^n$. To review:

Definitions (linear algebra in $\mathbb{R}^n$):

- The space of n-dimensional real vectors: $\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \cdots, x_n), \quad x_j \in \mathbb{R}\}$
- We can define an **inner product** (the 'dot product') on this space by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y} = \sum_{j=1}^{n} x_j y_j.$$

- This also defines a norm (the Euclidean or 'ℓ^2 norm')

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

- Two vectors $\mathbf{x}, \mathbf{y}$ are called **orthogonal** if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

  Geometrically: two vectors are orthogonal if they are perpendicular.

Some properties of the inner product and norm are worth highlighting:

- **Norm property:** A vector $\mathbf{x}$ has norm zero if and only if it is the zero vector:

$$\|\mathbf{x}\| = 0 \iff \mathbf{x} \equiv 0.$$

- **Linearity:** The inner product is **linear** in each argument; for the first:

$$\langle c_1\mathbf{u} + c_2\mathbf{v}, \mathbf{y}\rangle = c_1\langle \mathbf{u}, \mathbf{y}\rangle + c_2\langle \mathbf{v}, \mathbf{y}\rangle \text{ for all } c_1, c_2 \in \mathbb{R} \text{ and } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

and for the second argument,

$$\langle \mathbf{x}, c_1\mathbf{u} + c_2\mathbf{v}\rangle = c_1\langle \mathbf{x}, \mathbf{u}\rangle + c_2\langle \mathbf{x}, \mathbf{v}\rangle \text{ for all } c_1, c_2 \in \mathbb{R} \text{ and } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Operators: A **linear operator** L on $\mathbb{R}^n$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ (vectors to vectors) such that

$$L(c_1\mathbf{x} + c_2\mathbf{y}) = c_1 L\mathbf{x} + c_2 L\mathbf{y} \text{ for all } c_1, c_2 \in \mathbb{R} \text{ and } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

i.e. such that L is **linear**.

In $\mathbb{R}^n$, linear operators are equivalent to $n \times n$ matrices:

$$L \text{ is a linear operator} \iff \text{there is an } n \times n \text{ matrix } A \text{ s.t. } L\mathbf{x} = A\mathbf{x}.$$

1.1 Orthogonal bases

Recall that a set $\{\phi_1, \cdots, \phi_n\}$ is a **basis** for $\mathbb{R}^n$ if it (minimally) spans $\mathbb{R}^n$:

$$\text{every } \mathbf{v} \in \mathbb{R}^n \text{ has the form } \mathbf{v} = \sum_{j=1}^{n} c_j \phi_j \text{ for unique coefficients } c_j.$$

A set of vectors $\{\phi_j\}$ is said to be **orthogonal** if

$$\langle \phi_j, \phi_k \rangle = 0 \text{ for } j \neq k.$$

and, of course, an **orthogonal basis** is a basis that is orthogonal. Why are orthogonal bases so useful? The idea is that they separate the space into 'independent' parts that do not interact.

Let $\phi_1, \cdots, \phi_n$ be an orthogonal basis for $\mathbb{R}^n$. We know that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = \sum_{i=1}^{n} c_i \phi_i \text{ for coefficients } c_i.$$

To obtain the j-th coefficient c_j, we take the dot product $\langle \cdot, \phi_j \rangle$ with both sides. By orthogonality, all but one of the terms in the sum will cancel:

$$\begin{aligned}
\langle \mathbf{x}, \phi_j \rangle &= \sum_{i=1}^{n} c_i \langle \phi_i, \phi_j \rangle \\
&= \sum_{i \neq j}^{n} c_i \langle \phi_i, \phi_j \rangle + c_j \langle \phi_j, \phi_j \rangle \\
&= \sum_{i \neq j} c_i \cdot 0 + c_j \langle \phi_j, \phi_j \rangle \\
\implies & \boxed{c_j = \frac{\langle \mathbf{x}, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}.}
\end{aligned}$$

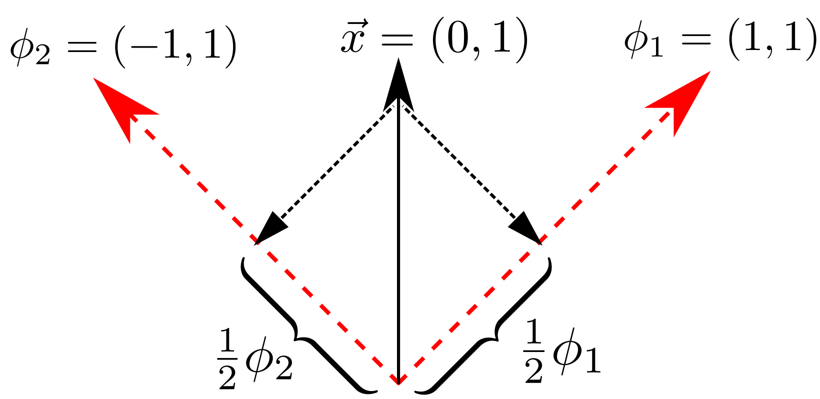
Due to orthogonality, **the equation for each c_j is independent of the others**.

What we are doing here is **projecting $\mathbf{x}$** onto its j-th component. The map

$$\mathbf{x} \to \frac{\langle \mathbf{x}, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}$$

extracts the coefficient c_j of the j-th component.

Observe, crucially, that this projection turns the n-dimensional system we need to solve for the c's into n one-dimensional systems (just scalar equations)!



In the example above, the basis vectors are $(\pm 1, 1)$ and

$$\mathbf{x} = c_1\phi_1 + c_2\phi_2 \implies c_1 = \frac{\langle \mathbf{x}, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \frac{1}{1^2 + 1^2} = \frac{1}{2}, \quad c_2 = \frac{\langle \mathbf{x}, \phi_2 \rangle}{\langle \phi_2, \phi_2 \rangle} = \frac{1}{2}.$$

1.2 Orthogonal bases of eigenvectors

Now, we get to the essential part - further improving the properties of the orthogonal basis.

An **eigenvalue** and associated **eigenvector** of a matrix A is a (possibly complex) number λ and vector $\phi \in \mathbb{R}^n$ such that

$$A\phi = \lambda\phi.$$

Given a matrix A, there is a particularly nice orthogonal basis, at least in a special case. Recall that a matrix is **symmetric** if $A^T = A$ and an important theorem:

Theorem (spectral theorem; matrices): Let A be an $n \times n$ real symmetric matrix. Then

- The eigenvalues $\lambda_1, \cdots, \lambda_n$ of A are real and distinct
- The corresponding eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are an orthogonal basis for $\mathbb{R}^n$.

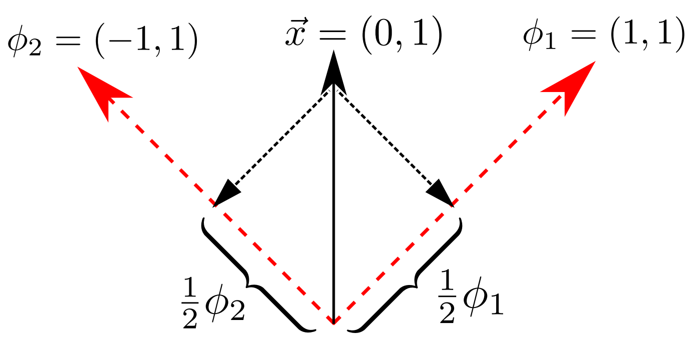
From here, we can interpret a matrix A as a linear operator that **scales components of its input along each eigenvector**. For instance, consider

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad \lambda_1 = 2,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then the operator

$$\mathbf{x} \to A\mathbf{x}$$

scales the $\mathbf{v}_1$ component by 2 and the $\mathbf{v}_2$ component by -2. Critically, applying the operator to an eigenvector yields a vector in **the same direction**.



It follows from this property that an orthogonal basis of eigenvectors 'diagonalizes' systems

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

That is, in the eigenvector basis, the linear system 'decouples' into n **independent (scalar) equations** that are trivial to solve.

Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be the eigenvectors of the symmetric matrix A. We solve the system (1) by projecting onto the j-th eigenvector as follows:

First, decompose $\mathbf{x}$ and $\mathbf{y}$ into its components in the eigenvector basis:

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{v}_j, \quad \mathbf{y} = \sum_{j=1}^{n} y_j \mathbf{v}_j.$$

Now plug into the equation and use the fact that $\mathbf{v}_j$ is an eigenvector ($A\mathbf{v}_j = \lambda_j \mathbf{v}_j$):

$$\begin{aligned} A\mathbf{x} &= y \\ A\left(\sum_{j=1}^{n} x_j \mathbf{v}_j\right) &= \sum_{j=1}^{n} y_j \mathbf{v}_j \\ \sum_{j=1}^{n} \lambda_j x_j \mathbf{v}_j &= \sum_{j=1}^{n} y_j \mathbf{v}_j \end{aligned}$$

When A is applied, each component **stays in the same direction** due to the eigenvector property, so the components stay independent of each other. Now use the fact that $\{\mathbf{v}_j\}$ is a basis to conclude that each component is equal:

$$\sum_{j=1}^{n} (\lambda_j x_j - y_j)\mathbf{v}_j = 0 \implies x_j = \frac{y_j}{\lambda_j} \text{ for } j = 1, \cdots, n.$$

Useful argument: Note that if $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is any basis and two vectors are equal, then each component is equal on its own:

$$\sum_{j=1}^{n} a_j \mathbf{v}_j = \sum_{j=1}^{n} b_j \mathbf{v}_j \implies a_j = b_j \text{ for all } j.$$

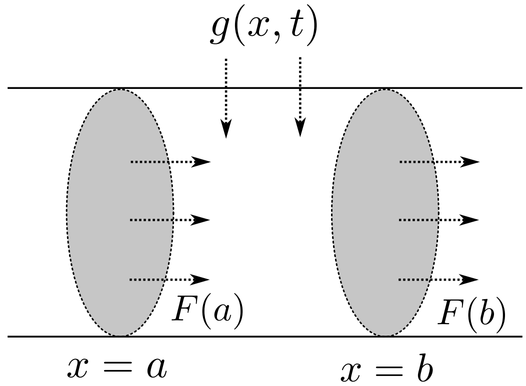
This is because 0 is uniquely represented by $0 = \sum_{j=1}^{n} 0 \cdot \mathbf{v}_j$, so

$$\sum_{j=1}^{n} (a_j - b_j)\mathbf{v}_j = 0 \implies a_j - b_j = 0 \text{ for all } j.$$

The structure exploited here will also appear in solving linear partial differential equations - and in fact, the calculations are similar (with basis vectors replaced by basis functions). To get there, of course, we must identify the right analogies to vectors, operators, bases etc. for functions, which will take some work.

2 Some context: PDEs from conservation laws

Rather than pull the equation out of thin air, let's see how PDEs arise naturally out of fundamental models[1]. To do so, we introduce the concept of a **conservation law**, which is a way of stating that for an amount of stuff in a region, the change in the amount is due to stuff entering/exiting the region or being created/destroyed. For simplicity, assume the stuff is 'heat' - but this argument is quite general (e.g. could be particle concentration, momentum, energy, density of fish, etc.)



Consider a cylindrical tube with cross section A running along the x-direction and $u(x,t)$ the temperature at position x and time t. The amount of heat in a section of the tube for x in some interval $[a,b]$ is

$$\int_a^b u(x,t)\,A\,dx.$$

Let us further suppose there is a **source** $g(x,t)$ that is the rate at which u is created or destroyed at position x along the tube. For instance, heat could leak out of the pipe at a rate $g(x,t)$ if the pipe is poorly insulated.

Define $F(x,t)$ to be the **flux** of heat: the rate at which heat flows through the cross section at x, with units of heat per (area)(time). Thus $\phi A\,dt$ is the amount of heat passing through the cross section in a time dt (with sign determining the direction). We have

$$\underbrace{\frac{\partial}{\partial t}\left(\int_a^b u(x,t)A\,dx\right)}_{\text{change in heat}} = \underbrace{AF(a,t) - AF(b,t)}_{\text{heat entering the section from the ends}} + \underbrace{\int_a^b g(x,t)A\,dx}_{\text{heat created/lost due to source}} .$$

Cancel out A and move the derivative on the LHS inside the integral, leading to

$$\int_a^b u_t(x,t)\,dx = F(a,t) - F(b,t) + \int_a^b g(x,t)\,dx,$$

which is a mathematical description of the conservation of heat.

[1]Adapted from *Applied Partial Differential Equations*, J. David Logan

Now write the F terms in an integral using the Fundamental Theorem of Calculus and collect all the terms to get

$$\int_a^b [u_t(x,t) + F_x(x,t) - g(x,t)]\, dx = 0.$$

The above equation must hold for **all** intervals $[a, b]$. It follows that the integrand must be equal to zero, leading to the 'differential form' of the conservation law,

$$u_t + F_x = g(x,t).$$

Many models in the sciences arise from this basic conservation argument. The next step is to determine the flux ϕ as as function of u and x (and the source).

Deriving the heat equation

If u is temperature, then the flux can be modeled by **Fourier's law**

$$\phi = -\alpha u_x$$

where α is a constant (the thermal diffusivity, with units of m^2/s). This simple law states that the the flux of heat is towards cooler areas, and the rate is proportional not to the amount of heat but to the gradient in temperature, i.e. the heat will flow faster if there is a large difference (e.g. an ice cube melting in a fridge vs. outside on a hot day).

Thus if there is no external source of heat, then u satisfies the **heat equation**

$$u_t = \alpha u_{xx}.$$

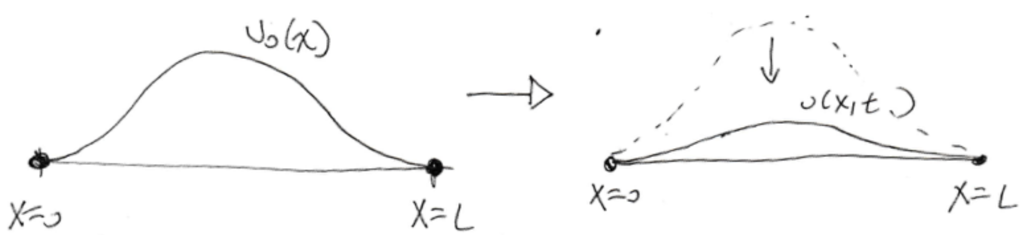
More generally, if u is any quantity whose flux is proportional to minus the gradient of u, then u will also satisfy the above. Such a process is called a **diffusion** process and the equation is then referred to as a **diffusion equation**.

3 Where are we going with all this?

4 Motivating example: Heat conduction in a metal bar

A metal bar with length $L = \pi$ is initially heated to a temperature of $u_0(x)$. The temperature distribution in the bar is $u(x,t)$. At the ends, it is exposed to air; the temperature outside is constant, so we require that $u = 0$ at the endpoints of the bar.

Over time, we expect the heat to diffuse or be lost to the environment until the temperature of the bar is in equilibrium with the air ($u \to 0$).



Physicist Joseph Fourier, around 1800, studied this problem and in doing so drew attention to a novel technique that has since become one of the cornerstones of applied mathematics. The approach outlined below hints at some of the deep structure we will uncover.

The temperature is modeled by the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad t > 0 \text{ and } x \in (0, \pi).$$

Since the temperature is fixed at both ends, we have

$$u(0,t) = 0, \quad u(\pi, t) = 0 \text{ for all } t.$$

Lastly, the initial heat distribution is $t = 0$ is

$$u(x, 0) = f(x)$$

where $f(x)$ is some positive function that is zero at 0 and π. The temperature should decrease as heat leaks out of the bar through the ends; eventually it all dissipates. In summary, we seek a function $u(x,t)$ defined on $[0, \pi]$ satisfying

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad t > 0 \text{ and } x \in (0, \pi), \tag{2a}$$

$$u(0,t) = u(\pi, t) = 0 \text{ for } t \geq 0 \tag{2b}$$

$$u(x, 0) = u_0(x). \tag{2c}$$

First, guess a solution of the form

$$u = e^{-\lambda t} \phi(x). \tag{3}$$

Substituting into the PDE (2a), we find that

$$-\lambda\phi(x) = \phi''(x).$$

Now substitute into the boundary conditions (2b) (note that $e^{-\lambda t}$ cancels out here) to get

$$\phi(0) = 0, \quad \phi(\pi) = 0.$$

It follows that (3), our guess for u, satisfies the PDE (2a) and the boundary conditions (2b) if the function $g(x)$ solves the **boundary value problem**

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = 0, \, \phi(\pi) = 0. \tag{4}$$

Though not an initial value problem, we can solve it by obtaining the general solution first. One can check that there are no solutions for $\lambda \le 0$. When $\lambda > 0$ the solution has the form

$$\phi = c_1 \sin(\mu x) + c_2 \cos(\mu x), \qquad \mu := \sqrt{\lambda}.$$

Imposing the conditions at 0 and 1 we find

$$\begin{aligned} \phi(0) = 0 &\implies \phi = c_1 \sin(\mu x) \\ \phi(1) = 0 &\implies \sin(\mu\pi) = 0. \end{aligned}$$

The second equation tells us that when μ is an integer a solution exists. This gives an infinite sequence of solutions to (4):

$$\lambda_n = n^2, \; \phi_n(x) = \sin(nx), \qquad n = 1, 2, 3, \cdots$$

It follows that the function

$$a_n e^{-n^2 t} \phi_n(x) \tag{5}$$

is a solution to the heat conduction problem except the initial condition. This 'one term' solution solves the problem with initial data

$$u_0(x) = a_n \sin(nx).$$

Now the crucial question: *what happens when the initial data is not a sine*? For systems of ODEs, we found a basis of n solutions whose span gave all solutions. Similarly, we must seek a solution that is an **infinite** linear combination of the one term solutions (5):

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \phi_n(x).$$

Then $u(x,t)$ solves the original problem (2) if the coefficients a_n satisfy

$$u_0(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{6}$$

Essentially, this is a representation of the function $u_0(x)$ in terms of the 'basis' *functions* $\{\sin(nx) : n = 1, 2, 3, \cdots\}$. In fact, this set has the remarkable **orthogonality** property

$$\int_0^\pi \phi_m(x)\phi_n(x)\,dx = \int_0^\pi \sin(mx)\sin(nx)\,dx = 0, \qquad m \neq n. \tag{7}$$

To solve for the coefficient a_m, we can multiply (6) by $\sin(mx)$ and integrate:

$$\int_0^\pi u_0(x)\sin(mx)\,dx = \int_0^\pi \sum_{n=1}^\infty a_n \sin(mx)\sin(nx)\,dx.$$

Now move the integral inside the sum (ignore rigor for now!). By the orthogonality property (7), only one of the terms in the sum will be non-zero:

$$\begin{aligned}
\int_0^\pi u_0(x)\sin(mx)\,dx &= \int_0^\pi \sum_{n=1}^\infty a_n \sin(mx)\sin(nx)\,dx \\
&= \sum_{n=1}^\infty a_n \int_0^\pi \sin(mx)\sin(nx)\,dx \\
&= \left(\sum_{n\neq m} a_n \cdot 0\right) + a_m \int_0^\pi \sin(mx)\sin(mx)\,dx \\
&= a_m \int_0^\pi \sin^2(mx)\,dx.
\end{aligned}$$

Miraculously, the infinite sum has been reduced to a simple equation for a_m:

$$a_m = \frac{\int_0^\pi u_0(x)\sin(mx)\,dx}{\int_0^\pi \sin^2(mx)\,dx}. \tag{8}$$

This process works for all m, so the solution to the heat conduction problem (5) with arbitrary initial condition $u_0(x)$ is

$$u(x,t) = \sum_{n=1}^\infty a_n e^{-n^2 t}\sin(nx)$$

with the coefficients given by the formula (8). The informal calculations here suggest a deeper structure, which will be the focus of our study: the properties and construction of this convenient basis; the consequences of this infinite series form and how it is used to reduce solving PDEs to simpler problems.

4.1 Where are we going with all this?

Ignoring all the rigorous details, let's identify the general idea. This example is intended to motivate some of the key questions. Consider a **partial** differential equation like

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in (a,b), \quad t > 0$$

for a function $u(x,t)$ and define the linear operator

$$Lu = \frac{\partial^2 u}{\partial x^2}.$$

Suppose L has eigenfunctions ϕ_j that form an orthogonal basis. Then we can express the solution in terms of this basis:

$$u(x,t) = \sum_{j=1}^{\infty} c_j(t)\phi_j(x)$$

for coefficients $c_j(t)$. Plug into the PDE:

$$\sum_{j=1}^{\infty} c_j'(t)\phi_j(x) = \sum_{j=1}^{\infty} \lambda_j c_j(t)\phi_j(x)$$

from which it (should) follow that

$$c_j' = \lambda_j c_j.$$

The orthogonal basis of eigenfunctions allows us to convert the (complicated) **PDE** into a set of (simple) one dimensional **ODEs** for the coefficients.

The Big Picture: The sketch above and the contrast between linear algebra in $\mathbb{R}^n$ and functions in L^2 raises some key questions that will motivate the topics to come. There are some equivalences, and many questions left to answer:

$$\begin{aligned}
\text{vectors in } \mathbb{R}^n (\text{ or } \mathbb{C}^n) &\iff \text{functions in ???} \\
\text{linear systems } Ax = b &\iff \text{linear DEs?} \\
\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i &\iff \langle f, g \rangle = \int_a^b f(x)g(x)\,dx \\
n \times n \text{ matrices} &\iff \text{linear operators } L \text{ (e.g. } d^2/dx^2, \cdots)
\end{aligned}$$

- **What is the operator?** We want an orthogonal basis of eigenvectors for some linear operator L. This means identifying the right operator and understanding when it will do what we want.

- **Infinite dimensions?** The basis for the function space is infinite dimensional - this has consequences that make the story more complicated than linear systems in $\mathbb{R}^n$.

- **What are the eigenfunctions?** We will need to study in detail how eigen-'functions' are different from eigenvalues (and some ways they are the same).

5 Linear algebra: function spaces

The first step leading up to solving linear PDEs is to identify the right space of functions and extend the idea of orthogonality from linear algebra in $\mathbb{R}^n$ to this space.

Consider **real** functions defined on an interval $[a, b]$. Define the **inner product**

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx \tag{9}$$

and call two functions f, g **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

Analogous to the Euclidean norm for a vector $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$$

we define the L^2 norm

$$\|f\|_2 = \left(\int_a^b |f(x)|^2\,dx \right)^{1/2}.$$

Definition (L^2 spaces) Consider (real) funcftions defined on an interval $[a, b]$. The set

$$L^2[a, b] = \{f : [a, b] \to \mathbb{R} \text{ such that } \|f\|_2 < \infty\}$$

is called the space of L^2 functions (pronounced 'ell-two'), or sometimes 'square-integrable' functions. Equivalently, a function is in L^2 if the integral of its square is finite,

$$\int_a^b |f(x)|^2\,dx < \infty.$$

The space $L^2[a, b]$ is a **vector space** (linear combinations of functions in L^2 are also in L^2 and it has an **inner product**

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

which is well-defined for all $f, g \in L^2[a, b]$. (we call this the 'L^2 inner product').

The L^2 norm gives us a way to measure distance between two functions. The expression

$$\|f - g\|_2^2 = \int_a^b |f(x) - g(x)|^2\,dx \tag{10}$$

is a sort of weighted measure of the area between the curves $f(x)$ and $g(x)$ on the inteval $[a, b]$. This is analogous to the Euclidean distance for vectors:

$$\|\mathbf{x} - \mathbf{y}\|^2 = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$$

which is the actual distance in $\mathbb{R}^n$ between the points at $\mathbf{x}$ and $\mathbf{y}$. The quantity (10) is sometimes called the **mean-square distance**.

Warning (complex functions): All the definitions here are true only for **real-valued functions**. For complex-valued functions, the inner product is instead

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

where $\overline{g(x)}$ is thet complex conjugate of $g(x)$. Most of the theory is the same, other than the occasional conjugate.

5.1 Some examples

For functions f and g, notions like orthogonality **depend on the underlying space** where the functions live. For instance, consider

$$f(x) = 1, \quad g(x) = \cos x.$$

Regarded as functions in $L^2[0,\pi]$, the two functions are orthogonal:

$$\langle f, g\rangle = \int_0^{\pi} \cos x\, dx = \sin x\Big|_0^{\pi} = 0.$$

However, as functions in $L^2[0,\pi/2]$ the two functions are not orthogonal, since then

$$\langle f, g\rangle = \int_0^{\pi/2} \cos x\, dx = 1.$$

The functions are orthogonal on $[0,\pi]$ but not on $[0,\pi/2]$; the domain matters because the definition of the inner product is different for each.

Another example: Consider the space $L^2[-1,1]$. We have that

$$\langle 1, x\rangle = \int_{-1}^{1} x\, dx = 0$$

so the constant function 1 and x are orthogonal on $[-1,1]$. However,

$$\langle 1, x^2\rangle = \int_{-1}^{1} x^2\, dx = \frac{2}{3},$$

so 1 and x^2 are not orthogonal. On the other hand, for $g(x) = x^2 - 1/3$,

$$\langle 1, g\rangle = \int_{-1}^{1} (x^2 - 1/3)\, dx = \frac{2}{3} - \frac{2}{3} = 0.$$

This means that the set

$$\{1, x, x^2 - 1/3\}$$

is an orthogonal set in $L^2[-1,1]$, whereas $\{1, x, x^2\}$ is not. The process, incidentally, can be continued to generate an orthogonal sequence of polynomials.