Math 353 Lecture Notes
Power Series Solutions

J. Wong (Fall 2020)

Topics covered

- Review of power series:
  - Basic properties, calculations with power series
  - Radius of convergence
- Series solutions (2nd order linear ODEs)
  - Motivation
  - Process for computing power series solutions
  - Simplifying the process ($\sum_{n=-\infty}^{\infty}$)
  - General solution / basis

1 Introduction

Earlier, we showed that solutions to homogeneous linear ODEs have the form

$$y = c_1\phi_1 + c_2\phi_2$$

where $\{\phi_1, \phi_2\}$ is a basis for the solution space. The problem of 'solving' the ODE was reduced finding a pair of coefficients, at the cost of having to obtain the basis functions - which might be complicated or impossible to find exactly.

Instead, a more robust approach might be to choose the basis functions ourselves. To this end, we might try a solution in the form of an infinite series:

$$y = \sum_{n=0}^{\infty} a_n\phi_n$$

where the ϕ_n's are not solutions, but simple functions we choose. By representing functions in this way, we can reduce the problem of solving an ODE to a sequence of equations for the coefficients. Of course, we now have to deal with (potentially) an infinite number of coefficients, and choose the functions ϕ_k in exactly the right way.

One possible choice for the ϕ_n's are polynomials, which leads to **power series solutions**. In this section, we seek solutions to linear ODEs like

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

by representing the solution by a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The disadvantages of the method are that solutions will be useful only near x_0 where the sum converges, and that the solution is an infinite series. On the other hand, the process is straightforward when it works.

2 Power series: Definitions

Convergence of series: Let $c_0, c_1, \cdots$ be a sequence of numbers. The series

$$\sum_{n=0}^{\infty} c_n$$

is said to be **convergent** if

$$\lim_{m\to\infty} \sum_{n=0}^{m} c_n \text{ exists}$$

and is said to be **absolutely** convergent if

$$\lim_{m\to\infty} \sum_{n=0}^{m} |c_n| \text{ exists.}$$

The **ratio test** says that if

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1 \tag{1}$$

then the series converges absolutely.

We will not need to worry about the difference between the two definitions of convergence; absolute convergence will be the relevant definition.

Definition: A **power series** centered at x_0 is a series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{2}$$

The series is well-defined only for x within some distance from x_0.The **radius of convergence** ρ is the largest value such that

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for all } x \text{ such that } |x-x_0| < \rho.$$

Within its radius of convergence, a power series converges and is therefore a function:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \text{defined for } |x-x_0| < \rho.$$

Typically, the ratio test is enough to find the radius of convergence. For instance, consider

$$\ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n.$$

Taking the ratio of successive terms, we find that

$$\lim_{n\to\infty} \frac{(-1)^{n+1}x^{n+1}/(n+1)}{(-1)^n x^n/n} = \lim_{n\to\infty} \frac{n}{n+1} x = x.$$

The ratio test (1) says that the series converges for x if the limit is less than one (in absolute value), so the series converges for $|x| < 1$. The radius of convergence is 1.

Note that one should take the ratio of successive non-zero terms, e.g. for

$$\sum_{n=0}^{\infty} \frac{1}{3^n} x^{2n},$$

$$\lim_{n\to\infty} \frac{x^{2n+2}/3^{n+1}}{x^{2n}/3^n} = \lim_{n\to\infty} \frac{1}{3}x^2 = \frac{1}{3}x^2 \implies \rho = 1/\sqrt{3}.$$

The radius of convergence can be zero! For instance, $\sum_{n=0}^{\infty} n! x^n$ converges only at $x = 0$.

The value of the power series (for our purposes) is that it provides 'local' approximations to a function near x_0. For instance, suppose we have a power series for $f(x)$ around zero:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

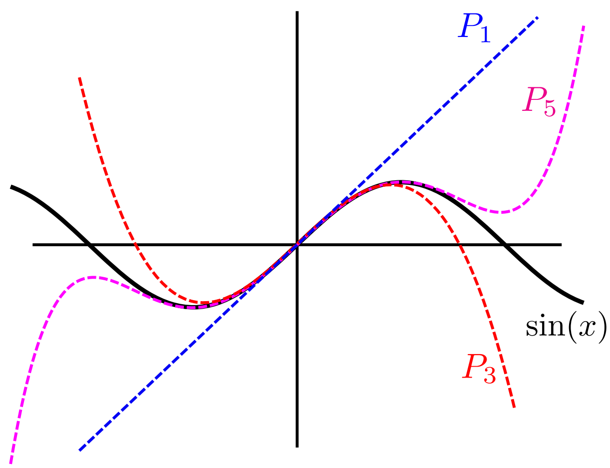
Define, for $m = 0, 1, 2, \cdots$,

$$P_m(x) = \sum_{n=0}^{m} a_n x^n$$

(the series with terms only up to degree m). Each P_m is a **partial sum** of the series[1]. For instance, for $\sin x$, the first few partial sums for the series around zero are

$$P_1 = x, \quad P_3 = x - \frac{x^3}{6}, \quad P_5 = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

Adding more terms improves the approximation, but all are accurate only near $x = 0$.



3 Calculations with power series

Power series have many convenient properties. The properties below show how they can be manipulated term by term. Here, a 'term' refers to a term of the sum, i.e. a coefficient times a power of x. We will often refer to 'the x^j term' to mean the term containing x^j.

For simplicity, we will assume the power series is around $x_0 = 0$ (but the results hold for any x_0 by just replacinh x^n with $(x - x_0)^n$).

Important technical note: formulas hereafter are valid for x in the radius of convergence of both series; this will be implied and not stated for each formula (the formulas are correct where they are defined).

Equality

Two power series are equal,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

if all the terms are equal. The equality above holds, for instance, if $a_n = b_n$ for all n.

[1]An ambiguity: there are several ways to count the number of terms. The 'm-th partial sum' may refer to the first m non-zero terms, the first m terms (including zero terms) or the terms up to degree m.

Addition

Power series can be added term-wise (by each power of x):

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

Often the indexing is different for two series. To add, align powers of x in each sum. For example,suppose we need to compute

$$\sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} b_n x^n.$$

Terms with the same power of x must be added together, so we need to shift the index to be $n+1$ instead of n. In full detail, here is the calculation:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^{n+1} \\
&= \sum_{n=0}^{\infty} a_n x^n + \sum_{m=1}^{\infty} b_{m-1} x^m \qquad (\text{set } m = n+1) \\
&= \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_{n-1} x^n \qquad (\text{relabel } m \text{ as } n) \\
&= a_0 + \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_{n-1} x^n \qquad (\text{match starting index}) \\
&= a_0 + \sum_{n=1}^{\infty} (a_n + b_{n-1}) x^n.
\end{aligned}$$

With the first few terms written out:

$$\begin{aligned}
& a_0 + \quad a_1 x + \quad a_2 x^2 + \cdots \\
& \qquad\qquad b_0 x + \quad b_1 x^2 + \cdots \\
= \; & a_0 + (a_1 + b_0) x + (a_2 + b_1) x^2 + \cdots
\end{aligned}$$

The second sum is written with a new index $m = n+1$ so that the power of x is x^m, matching the first sum. Then the two series can be added term by term. Note that the x^0 term in the first sum is alone, since the second sum starts at x^1.

The explicit change of index to m (and then back again) is often skipped. A terser version of the calculation would look like the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^{n+1} &= \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_{n-1} x^n \\
&= a_0 + \sum_{n=1}^{\infty} (a_n + b_{n-1}) x^n.
\end{aligned}$$

Multiplication

The product of two power series

$$\sum_{n=0}^{\infty} a_n x^n, \quad \sum_{n=0}^{\infty} b_n x^n$$

is itself a power series:

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right).$$

The right hand side will be a sum of all pairwise products of terms in the first sum with terms in the second sum. To find the n-th coefficient c_n, collect terms that multiply to give x^n. The terms $a_j x^j$ and $b_{n-j}x^{n-j}$ will combine to give $a_j b_{n-j} x^n$. Writing things out:

$$(a_0+a_1x+a_2x^2+\cdots)(b_0+b_1x+b_2x^2+\cdots) = a_0b_0+(a_0b_1+a_1b_0)x+(a_2b_0+a_1b_1+a_0b_2)x^2+\cdots$$

The result is that

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

The most important special case is when one power series is simple, in which case the formula does not need to be invoked. For instance,

$$x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

If there are a few terms, it is not too bad to compute by computing each piece separately:

$$\begin{aligned}(x^2-3x)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} 3a_n x^{n+1} \\ &= \sum_{n=2}^{\infty} a_{n-2} x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n \\ &= 3a_0 x + \sum_{n=2}^{\infty} (a_{n-2} - 3a_{n-1}) x^n.\end{aligned}$$

Note that the first sum is shifted by 2; the second one is shifted by 1. You could, of course, only shift one of them by one instead and end up with x^{n+1} or x^{n+2} in the result.

Differentiation

Within its radius of convergence, a power series may be differentiated any number of times and this can be done termwise:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Note that since the a_0 term is a constant, it vanishes when taking the derivative. The radius of convergence of all the derivatives is the same as the original series.

4 A useful trick (series arithmetic, the easy way)

Note that the 'extra' terms (like $3a_0x$ above) are exceptions that must be outside the sum. An alternate approach simplifies this process (and is suggested for calculations, if you want).

Instead, we write a power series as a 'doubly infinite' sum:

$$\sum_{n=0}^{\infty} a_n x^n \to \sum_{n=-\infty}^{\infty} a_n x^n$$

where we have **extended** the coefficients by zeros:

$$a_n = 0 \text{ for } n < 0.$$

Now, there are no exceptional terms! To illustrate by example:

Addition of series: Take the example from before,

$$\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} b_n x^n.$$

Extend both coefficients by zeros:

$$a_n, b_n = 0 \text{ for } n < 0.$$

Now index-shifting the second term is easier. Below, **all sums are from** $-\infty$ to ∞ (for 'all integers n').

$$\begin{aligned} x\sum_{n=-\infty}^{\infty} b_n x^n &= \sum_n b_n x^{n+1} \\ &= \sum_n b_{n-1} x^n \quad (\text{shift } n \to n-1) \end{aligned}$$

Note that since the range is $-\infty$ to ∞, the range doe snot change!

The addition then looks like:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} b_n x^n. &= \sum_n a_n x^n + \sum_n b_n x^{n+1} \\ &= \sum_n a_n x^n + \sum_n b_{n-1} x^n \quad (\text{shift } n \to n-1) \\ &= \sum_n (a_n + b_{n-1}) x^n. \end{aligned}$$

We see that when $n \leq -1$, the coefficient is zero, leaving

$$(a_0 + b_{-1})x^0 + (a_1 + b_0)x^1 + \cdots$$

which gives $a_0x^0 + (a_1 + b_0)x^1$ as calculated before.

Differentiation: This method is useul here also, but remember to pay attention to terms that are zero. For example,

$$\frac{d^2}{dx^2}\sum_n a_n x^n = \sum_n a_n n(n-1)x^{n-2} \underbrace{=}_{\text{shift } n \to n+2} \sum_n a_{n+2}(n+2)(n+1)x^n.$$

Note that this sum (the non-zero part) starts at $n = 0$ since

$$a_{n+2}(n+2)(n+1) = 0 \text{ if } n = -1, n-2$$

because of the $(n+2), (n+1)$ factors, so this is really a power series

$$2a_2x^0 + 6a_3x^1 + \cdots .$$

4.1 Note on arbitrary coefficients

The missing coefficients matter! Consider the equation

$$y'' = 7.$$

If y is a power series

$$y(x) = \sum_n a_n x^n$$

then we get

$$\sum_n a_{n+2}(n+2)(n+1)x^n = 7 + \sum_{n\geq 1} 0 \cdot x^n.$$

Equating terms of each power, we get

$$\begin{aligned} x^n \text{ term } &: \quad a_{n+2} = 0 \text{ for } n \geq 1 \\ x^0 \text{ term } &: \quad 2a_2 = 7 \\ x^{-1} \text{ term } &: \quad 0 \cdot a_1 = 0 \\ x^{-2} \text{ term } &: \quad 0 \cdot a_0 = 0 \end{aligned}$$

Thus there are two **arbitrary coefficients**, and we get

$$y(x) = \frac{7}{2}x^2 + a_1x + a_0, \qquad \text{for any } a_0, a_1 \in \mathbb{R}.$$

5 More on combining power series

Taylor series

Suppose
$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$
Differentiating k times, we find that
$$f^{(k)}(x) = \frac{d^k}{dx^k}\left(\sum_{n=0}^{\infty} a_n x^n\right) = k! a_k + \cdots$$
The first $k-1$ terms vanish; all the others in the ellipsis have an x in them. Evaluating at zero, we get
$$f^{(k)}(0) = k! a_k$$
which gives the familiar formula for the Taylor series,
$$a_k = \frac{1}{k!} f^{(k)}(0). \tag{3}$$
Thus we can easily compute power series for functions when we know their derivatives.

For example, if we know $f(x)$ has a power series
$$f(x) = x + \frac{1}{6}x^4 + \cdots$$
Then $f'(x) = 1 + \frac{2}{3}x^3 + \cdots$ so $f'(0) = 1$. From this direct calculation (differentiating more times) or Taylor's formula we can compute
$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = f'''(0) = 0, \quad f^{(4)}(0) = 4! \cdot \frac{1}{6} = 4.$$
Conversely, of course, we can get the power series for known functions using Taylor's formula (3), e.g.
$$f(x) = \frac{1}{x-1} \implies f^{(n)}(x) = (-1)^n \frac{n!}{(x-1)^{n+1}} \implies f^{(n)}(0) = -n!$$
so Taylor's formula yields the familiar geometric series
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

5.1 Combining series, examples

We will need to use these rules to evaluate expressions involving power series. It is important to make sure to align powers of x and deal with 'stray' terms in one sum but not the other. In particular, note that differentiating removes terms and tends to shift the index (depending on how you write it).

Some examples will illustrate the typical calculations.

Example 1: Suppose
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
and we want to compute the series centered at $x = 0$ for
$$f''(x) - xf(x).$$
Differentiating term-wise and then aligning indices, we get
$$\begin{aligned} f''(x) - xf(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\ &= 2a_2 + \sum_{n=1}^{\infty} ((n+2)(n+1)a_{n+2} - a_{n-1}). \end{aligned}$$
For the second step, the first index is shifted so that $n-2$ becomes n ($m = n-2$) and the second index is shifted so that $n+1$ becomes n ($m = n+1$). We'll use this result shortly to solve an ODE (see next section).

Example 2: Suppose
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
and we want to compute the series centered at $x = 0$ for
$$g(x) = f''(x) + x^3 f'(x).$$
Plugging in the series for f and computing term-wise, we get
$$\begin{aligned} g &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^3 \sum_{n=1}^{\infty} a_n n x^{n-1} \\ &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n+2}. \end{aligned}$$

Note that when taking f'', we lose the x^0 and x^1 terms, so the indexing starts at 2. Now replace the index in the first term with $m = n-2$ and the index in the second term with

$m = n+2$ (relabeling the new index as n again) to combine:

$$\begin{aligned} g &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n+2}. \\ &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=3}^{\infty} a_{n-2}(n-2)x^n \\ &= 2a_2 + 6a_3 x + 12a_4 x^2 + \sum_{n=3}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_{n-2}(n-2)\right] x^n. \end{aligned}$$

Note that the second sum starts with the x^3 term, so the terms up to x^2 in the first sum do not get combined with anything in the second sum (so they get written separately)

It is worth noting that you can sanity check your results at any step by just writing out a few terms of each sum. For instance,

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = (\underbrace{a_2 \cdot 2 \cdot 1 \cdot x^0}_{n=2} + \underbrace{a_3 \cdot 3 \cdot 2 \cdot x^1}_{n=3} + \cdots$$

and the shifted version is

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = \underbrace{a_2 \cdot 2 \cdot 1x^0}_{n=0} + \underbrace{a_3 \cdot 3 \cdot 2 \cdot x^1}_{n=1} + \cdots$$

so they are indeed equal. Writing out a few terms catches most errors in computation (which are easy to make because of the bookkeeping involved).

5.2 Alternate method: starting with $-\infty$

The alternate approach greatly simplifies the process, and is recommended for these calculations. Consider Example 2 again; we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and wish to compute $f'' + x^3 f'$. Extending the a_n's by zeros, we have

$$f(x) = \sum_{-\infty}^{\infty} a_n x^n.$$

Write sums $\sum_n$ to mean $\sum_{n=-\infty}^{\infty}$. Then

$$f'(x) = \sum_n n a_n x^{n-1} \quad f''(x) = \sum_n n(n-1)a_n x^{n-2}$$

Continuing with the example, we get

$$\begin{aligned}
f''(x) + x^3 f'(x) &= \sum_n n(n-1)a_n x^{n-2} + \sum_n n a_n x^{n+2} \\
&= \sum_n (n+2)(n+1)a_{n+2}x^n + \sum_n (n-2)a_{n-2}x^n \\
&= \sum_n \left[(n+2)(n+1)a_{n+2} + (n-2)a_{n-2}\right] x^n \\
&= \sum_n b_n x^n
\end{aligned}$$

where

$$b_n = (n+2)(n+1)a_{n+2} + (n-2)a_{n-2}.$$

You should check that the b_n's are zero for $n < 0$, so this result really is a power series as expected.

Now suppose we want to solve the ODE

$$f'' + x^3 f' = 0.$$

Then

$$\sum_n b_n x^n = \sum_n 0 \cdot x^n$$

$$\implies (n+2)(n+1)a_{n+2} + (n-2)a_{n-2} = 0 \text{ for all } n.$$

The first few relevant equations are:

$$n = -2 \implies 0 \cdot a_0 = 0$$

$$n = -1 \implies 0 \cdot a_1 = 0$$

$$n \geq 0 \implies a_{n+2} = \frac{-(n-2)}{(n+2)(n+1)} a_{n-2}.$$

This gives that

$$a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = -\frac{1}{20}a_1 \cdots$$

and so on. You can then check from the formula that only a_0 and the coefficients $a_1, a_5, a_9, \cdots$ are non-zero, and all multiples of a_1. This gives

$$f(x) = a_0 + a_1(x - \frac{1}{20}x^5 + c_9 x^9 + \cdots)$$

for coefficients $c_5, \cdots$ you could calculate. The result here makes sense because the ODE is a first order ODE for f', and the solution is

$$f = c_1 + c_2 \int e^{-x^4/4}\, dx$$

The process here suggests that we can use power series to get solutions to ODEs, the subject of the next section.

6 Series solutions: Basics

Our goal is to obtain local approximations to a linear, second order ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{4}$$

near $x = 0$.[2] In particular, we seek two linearly independent solutions as power series:

$$y_1 = \sum_{n=0} b_n x^n, \quad y_2 = \sum_{n=0}^{\infty} c_n x^n.$$

We will use the following straightforward process:

1) Assume $y(x)$ has the form of a power series:

$$y(x) = \sum_{n=0} a_n x^n. \tag{5}$$

2) Write P, Q and R as power series as well:

$$P = \sum_{n=0}^{\infty} p_n x^n, \quad Q = \sum_{n=0}^{\infty} q_n x^n, \quad R = \sum_{n=0}^{\infty} r_n x^n.$$

Most of the time, P, Q and R are simple (polynomials).

3) Substitute y into the ODE and use the multiplication/addition/differentiation rules to write the result as a single sum:

$$0 = \sum (\text{expression involving } a_n\text{'s}) x^n.$$

4) Conclude that the coefficient of x^n in the above must be zero for each n

5) Solve the equations to get the a_n's.

6) From the general solution (5), identify a solution basis and verify that they are linearly independent.

Practically, this means we can approximate solutions by a few terms, and then add more as needed to improve accuracy or understand finer properties of solutions. The process only works under certain conditions (which we address later).

[2] The methods herein work equally well for inhomogeneous ODEs, for higher order linear ODEs and solutions around other points x_0; we focus on (4) for simplicity as the general cases does not give any more insight into the idea.

6.1 Illustrative example:

To see how the method works, we find a series solution for

$$y'' + y = 0. \tag{6}$$

Get equations for coefficients: We assume a solution series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{7}$$

and substitute into the ODE:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now shift the index on the first term by two:

$$\sum_{n=0}^{\infty} (a_{n+2}(n+2)(n+1) + a_n)x^n = 0.$$

For this equation to be true, we need the coefficients of x^n to be zero for all n. Setting the coefficients to zero yields a **recurrence relation**:

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \qquad n = 0, 1, 2 \cdots . \tag{8}$$

Notice that there is no equation for a_0 or a_1. We have shown that any sequence satisfying (8) will make the series (7) a solution to the ODE (6). There is no restriction on a_0 or a_1; they are arbitrary.

Solve for the coefficients: To find the coefficients, we iterate the recurrence relation. It tells us that we solve for the a_n's as follows:

$$\begin{aligned} &a_0 \to a_2 \to a_4 \to a_6, \cdots \\ &a_1 \to a_3 \to a_5 \to a_7, \cdots \end{aligned}$$

For even coefficients, using the recurrence to get a_2 and a_4 gives

$$a_2 = -\frac{a_0}{1 \cdot 2} = -\frac{a_0}{2!}, \quad a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{4!}.$$

This suggests that

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}, \qquad n = 1, 2, \cdots \tag{9}$$

In general, for even indices,

$$a_{2n} = -\frac{a_{2n-2}}{2n(2n-1)}$$

and it is straightforward to show from this that our guess (9) is correct.

For the odd coefficients, we have

$$a_3 = -\frac{a_1}{3!}, \quad a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{5!}$$

which suggests that

$$a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}, \qquad n = 1, 2, \cdots . \tag{10}$$

Note that the odd coefficients are all multiples of a_1 and the even ones are all multiples of a_0.

We can therefore split the power series for y into two parts:

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}.$$

Now use the formulas (9) and (10) for the coeficients:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}.$$

This is in the form of the span of two solutions,

$$y = a_0 y_1 + a_1 y_2,$$

where

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x,$$

$$y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x$$

(by recognizing the power series for $\cos x$ and $\sin x$). The result makes sense since, of course, $\cos x, \sin x$ are a basis for solutions to $y'' + y = 0$.

Checking linear independence: Last, we need to verify that y_1, y_2 are linearly independent. This can be done from the series form. The first few terms of each are

$$y_1 = 1 - \frac{1}{2}x^2 + \cdots, \qquad y_2 = x - \frac{1}{6}x^3 + \cdots$$

From this we can easily compute

$$y_1(0) = 1, \quad y_1'(0) = -x + \cdots \Big|_{x=0} = 0$$

and

$$y_2(0) = 0, \quad y_2'(0) = 1 - \frac{1}{2}x^2 + \cdots \Big|_{x=0} = 1$$

so the two solutions are linearly independent. (You could also just use Taylor's theorem directly instead of computing $f'(x)$ then plugging in 0).

7 A more difficult example

We find the general series solution to the Airy equation

$$y'' = xy$$

which arises in various applications in physics (e.g. optics, quantum mechanics). Unlike the previous example, the Airy equation does not have an exact solution, so the series solution will give information we cannot otherwise obtain!

Again, look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Here, let's use the extension trick and write

$$y = \sum_{n=-\infty}^{\infty} a_n x^n, \qquad a_n = 0 \text{ for } n < 0.$$

Now substitute into the ODE:

$$\sum_n a_n n(n-1)x^{n-2} - \sum_n a_n x^{n+1} = 0.$$

Now shift the index of the first sum up by **three**:

$$\sum_n \left(a_{n+3}(n+3)(n+2) - a_n\right) x^{n+1} = 0.$$

Plugging in $n = -2$ and $n = -3$ we find that

$$0 \cdot a_0 = 0,\ 0 \cdot a_1 = 0 \implies a_0, a_1 \text{ arbitrary}.$$

Now for $n = -1$,

$$2a_2 = 0 \implies a_2 = 0$$

so a_2 must be **zero** (not arbitrary!). For the rest,

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}, \quad n \geq 0. \tag{11}$$

Since a_n depends on a_{n-3}, (11) produces three independent sequences of coefficients.

First, the one starting with a_2. Since $a_2 = 0$, we also have

$$0 = a_2 = a_5 = a_8 = \cdots$$

so $a_{3m+2} = 0$ for all m.

For the other two parts, observe that $a_3, a_6, a_9, \cdots$ are all multiples of a_0:

$$a_{3m} = c_{3m} a_0 \text{ for certain coefficients } c_{3m}$$

and similarly

$$a_{3m+1} = c_{3m+1} a_1 \text{ for certain coefficients } c_{3m+1}$$

Plugging all this into the series and breaking it up into the three parts (the zeros, the a_0 and a_1 parts), we find that

$$\begin{aligned} y(x) &= a_0 \sum_{m=0}^{\infty} c_{3m} x^{3m} + a_1 \sum_{m=0}^{\infty} c_{3m+1} x^{3m+1} \\ &= a_0 y_1(x) + a_1 y_2(x) \end{aligned}$$

which is exactly the desired structure. To verify linear independence, it's not hard to compute from the recurrence that the first two terms of y_1, y_2 are

$$y_1 = 1 + \frac{1}{6}x^3 + \cdots, \quad y_2 = x + \frac{1}{12}x^4 + \cdots$$

Since

$$y_1(0) = 1,\ y_1'(0) = 0, \qquad y_2(0) = 0,\ y_2'(0) = 1$$

the Wronskian is 1 at $x = 0$ so the solutions are indeed linearly independent.

Extra (formula for the coeffs): To compute the coefficients, it's convenient to rewrite

$$a_n = \frac{n-2}{n(n-1)(n-2)} a_{n-3}.$$

The reason is that the $n(n-1)(n-2)$ in the denominator means that we will get $n!$ in the denominator for a_n. The numerator will give factors $1 \cdot 4 \cdots$ and $2 \cdot 5 \cdots$ (each factor goes up by three each iteration). The result is that

$$a_{3m} = \frac{1 \cdot 4 \cdots (3m-2)}{(3m)!} a_0, \quad a_{3m+1} = \frac{2 \cdot 5 \cdots (3m-1)}{(3m+1)!} a_1.$$

Setting $a_0 = 1$ and $a_1 = 0$ we get a solution with only the x^{3m} terms:

$$y_1 = \sum_{m=0}^{\infty} \left(\frac{1 \cdot 4 \cdots (3m-2)}{(3m)!} \right) x^{3m}$$

and setting $a_0 = 0$ and $a_1 = 1$ we get a solution with only x^{3m+1} terms:

$$y_2 = \sum_{m=0}^{\infty} \left(\frac{2 \cdot 5 \cdots (3m-1)}{(3m+1)!} \right) x^{3m+1}.$$

The general solution is

$$y = a_0 y_1 + a_1 y_2.$$

8 Theory for series solutions: Ordinary points

We have found, for a few examples, power series solutions to **linear** equations

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

The main questions are:

- Under what conditions can we find a power series solution to the ODE?
- If such a solution exists, where is it valid (what is the radius of convergence?)
- What can we do if a power series does not exist?

8.1 Analytic functions

Definition: A function $f(x)$ is called **analytic**[3] at a point x_0 if it has a power series expansion at x_0 with a non-zero radius of convergence ρ:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ for } |x - x_0| < \rho.$$

Informally: a function is analytic at x_0 if it has a power series at x_0.

For some examples, consider, around $x = 0$,

$$\text{(i) } \sin x/x, \quad \text{(ii) } \frac{\cos x}{x}, \quad \text{(iii) } x^{5/2}$$

i) Analytic at $x = 0$ because it has a power series

$$\sin x/x = (x - x^3/6 + \cdots)/x = 1 - x^2/6 + \cdots$$

ii) Not analytic because it has a singularity at $x = 0$. Attempting to use the approach from (i) gives

$$\cos x/x = (1 - x^2/2 + \cdots)/x = \frac{1}{x} + \cdots$$

so there is a $1/x$-like singularity at $x = 0$.

iii) Not analytic at $x = 0$. To have a power series, the function must have derivatives of all orders at $x = 0$, but

$$(x^{5/2})''' = Cx^{-1/2}$$

which is infinite at $x = 0$.

[3]Technicality: This really should be 'real analytic' for a real-valued function, or the condition should be for *complex* numbers with $|z - z_0| < \rho$, but the distinction is irrelevant here.

8.2 Ordinary points (when do power series solutions exist?)

Whether the power series method can be used on a linear ODE depends on whether the coefficient functions are analytic. For simplicity, we consider second order ODEs of the form

$$y'' + p(x)y' + q(x)y = 0 \tag{12}$$

Definition: We call a point x_0...

- ...an **ordinary point** of the ODE (12) if p and q are analytic at x_0,
- ...a **singular point** of the ODE if either p or q fail to to be analytic x_0,

For example, the ODE

$$(x^2-1)^2y'' + (\sin x)y = 0$$

has an ordinary point at $x = 0$. In the 'standard form,

$$y'' + \frac{\sin x}{x^2-1}y = 0.$$

The coefficient of y' is zero and the coefficient of y is

$$q = \frac{\sin x}{(x^2-1)^2} = \frac{\sin x}{(x-1)^2(x+1)^2}$$

which has singularities at $x = \pm 1$, but is otherwise analytic. So in fact, is has ordinary points for all x_0 except $x_0 = \pm 1$.

(Note: the fact that the functions are analytic away from the singularity is known since $\sin x$, $1/(x\pm 1)^2$ have power series and so their product does as well).

As a second example, the *Bessel equation of order* α

$$x^2y'' + xy' + (x^2-\alpha^2)y = 0$$

(with α a constant) has a regular singular point at $x = 0$, since it is equivalently

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0.$$

8.3 A hint at what happens at singular points

A simple example illustrates what can go wrong at singular points. Consider

$$4x^2y'' + 4xy' - y = 0$$

We know this as an Euler equation; recall that plugging in x^r yields solutions when

$$0 = 4r^2 - 1 = (r-1/2)(r+1/2)$$

so the solution is

$$y(x) = c_1 x^{1/2} + c_2 x^{-1/2}$$

This ODE has a singular point at $x = 0$ since

$$y'' + \frac{1}{x}y' - \frac{1}{4x^2}y = 0$$

has singular coefficients there. Notice that the solution has no power series at $x = 0$ (because $x^{1/2}$) has no power series at $x = 0$), so the power series method cannot work.

You can check that looking for a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and plugging in via the usual procedure will **not** yield two linearly independent solutions!

For the details: Plug in and move the $1/x$, $1/x^2$ coefficients inside the sum to get

$$\sum_n a_n n(n-1)x^{n-2} + \sum_n a_n n x^{n-2} - \frac{1}{4}\sum_n a_n x^{n-2} = 0$$

which does not require any shifting! Combine to get

$$\sum_n a_n \Big(n(n-1) + n - \frac{1}{4}\Big)x^{n-2} = 0.$$

The equations to solve for the coefficients are then

$$(n^2 - \frac{1}{4})a_n = 0 \quad \text{for integers } n \geq 0.$$

But for the power series, the n's are integers so the factor $n^2 - 1/4$ is never zero, which yields

$$a_n = 0 \text{ for all } n \implies y(x) = 0.$$

THus, the only power series solution is the trivial one.

8.4 Main result: ordinary points

Near an ordinary point, the coefficients $p(x)$ and $q(x)$ in the ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{13}$$

can be expanded in a power series to obtain

$$y'' + \Big(\sum_{n=0} p_n (x-x_0)^n\Big)y' + \Big(\sum_{n=0} q_n (x-x_0)^n\Big)y = 0.$$

If we can plug in $y = \sum a_n(x-x_0)^n$, all the operations (derivatives, multiplication, addition) on power series yield another power series, so the result will be

$$\sum_{n=0}^{\infty}(\cdots)(x-x_0)^n = 0$$

where each coefficent of $(x-x_0)^n$ is an expression involving the unknowns $a_0, \cdots, a_n$, leading to a recurrence that can be solved. It turns out that the resulting recurrence will always have a solution at an ordinary point. The main result (not proven here) confirms this observation:

Theorem on ordinary points: (i) At an ordinary point x_0, The ODE (13) has a power series solution with a non-zero radius of convergence, which will have the form

$$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 y_1 + a_1 y_2$$

for linearly independent series solutions y_1, y_2.

(ii) Moreover, the series for y_1, y_2 are **guaranteed to converge** in the interval where the series for p and q both converge.

That is, the radius of convergence of y_1 (and y_2) is at least as large as the minimum of the radii of convergence for p and q.

The second part gives us a lower bound on the region where solutions will converge just by looking at the ODE. We simply need to find where p and q have convergent power series and take the intersection.

For example, the *Airy equation*

$$y'' = xy$$

has an ordinary point at every x_0 since x is a polynomial. Since

$$x = \sum_{n=0}^{\infty} a_n x^n, \quad a_1 = 1,\, a_n = 0 \text{ otherwise}$$

is already a power series, it converges everywhere (the 'sum' is just one term). Thus the series solutions to the Airy equation also converge everywhere.

On the other hand, consider a series around $x_0 = 0$ for

$$(1-x)y'' + xy' + y = 0. \tag{14}$$

The coefficients $p(x) = x/(1-x)$ and $q = 1/(1-x)$ both have series around zero with radius of convergence 1 (check this!) so any series solution $y = \sum a_n x^n$ to (14) has radius of convergence at least one. We do **not** know that the radius of convergence for a solution will be exactly one (it could be larger!).