Math 353 Lecture Notes
Dirac delta, impulse

J. Wong (Fall 2020)

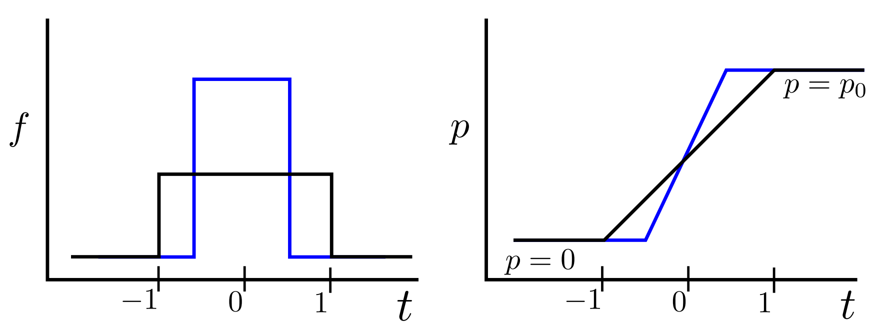
Topics covered

- Dirac delta and instantaneous forcing
  - Dirac delta properties; physical interpretation
  - Laplace transform of $\delta(t)$
  - Applications to ODEs
  - Connection to convolutions; impulse response
- Inverse transform examples
  - Suggestions for computations
  - Some miscellaneous rules
  - Resonance, revisited

1 Instantaneous forcing: a problem

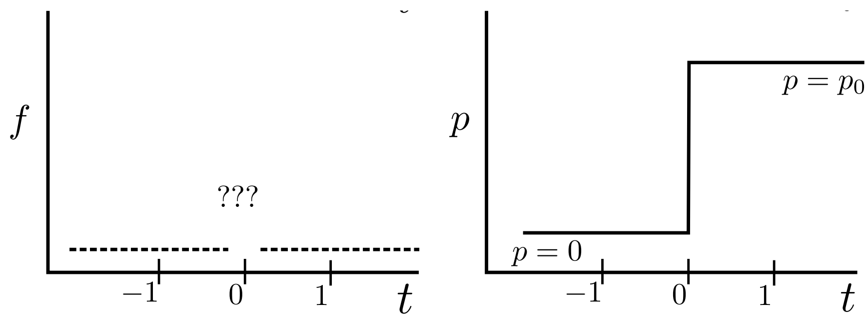
Let us consider a simple physics problem, which we will find has a major technical issue. Suppose a (frictionless) box is at rest. It is given a push (a force) over some finite time, leading to a momentum of p_0.

Suppose the force is applied from time $t = -\epsilon$ to time $t = \epsilon$. To give a total p_0 momentum, the force must have the value $p_0/2\epsilon$ (see $p(t)$ and $f(t)$ in the figure below).



When the forcing is active, the momentum increases linearly, and then is constant once the forcing is stopped. The linear increase from 0 to p_0 occurs over the interval $(-\epsilon, \epsilon)$ when the force is applied.

Now imagine that the push occurs instantaneously, adding a momentum of p_0 instantly at $t = 0$. Then the momentum vs. time graph is a step function.



What, then, is the correct $f(t)$? We know that because there is no force for $t \neq 0$, it must be that $f \neq 0$. From

$$\frac{dp}{dt} = f(t)$$

we know that the force must satisfy

$$\int f(t)\, dt = p_0.$$

All of the momentum (all the contribution to the integral) must be concentrated exactly at $t = 0$. However, there is no function $f(t)$ with this property!

Indeed, attempting to take the limit of forces that act over times $(-\epsilon, \epsilon)$ is equally futile, since the limit does not give a well defined function.

It is clear from this simple example that we need a new object to describe an instantaneous force that can make a solution 'jump' (have a discontinuity).

Another analogy: Suppose we have some distribution of mass on a line. If the (linear) density is $\rho(x)$ then the total amount of mass in an interval $[a, b]$ is

$$m = \int_a^b \rho(x)\, dx.$$

But what if we have a mass of 1, all concentrated at a single point $x = 0$? We would need to have a 'density' function that is zero except at $x = 0$ but

$$\int_{-\infty}^{\infty} \rho(x)\, dx = 1.$$

There is no 'density' function for a mass concentrated at a point! We need a new object to represent such a thing.

2 The dirac delta

Inspired by the example, let us define a sequence of box functions as follows:

$$g_\epsilon(t) = \frac{1}{2\epsilon}(u_\epsilon - u_{-\epsilon}) = \begin{cases} 1/2\epsilon & \text{if } -\epsilon < t < \epsilon \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

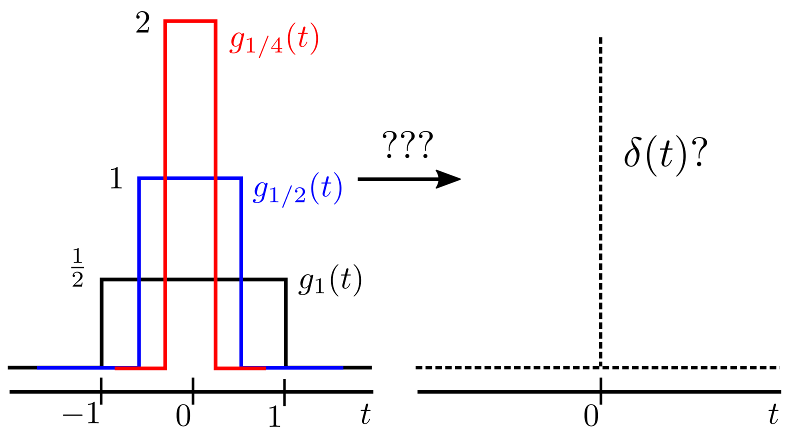
Each function has an integral of 1, i.e.

$$\int_{-\infty}^{\infty} g_\epsilon(t)\, dt = 1,$$

and the function is non-zero only in the small interval $(-\epsilon, \epsilon)$.

We would like to define a new object as a 'limit' of these boxes as $\epsilon \to 0$:

$$\delta = \lim_{\epsilon \to 0} ?\, g_\epsilon. \quad (2)$$



The question mark is there because the limit is not a well-defined function. Ignoring this issue, we shall call this object the **Dirac delta**. The key point is that although (2) is not defined, when inside other expressions, the limits can work out. So, very roughly,[1] the limit (2) is meant in the sense that

[expression with δ] **means** evaluate [expression] for g_ϵ in place of δ, then take $\epsilon \to 0$.

Typically, expression means something with an integral. For instance,

$$\int_{-\infty}^{\infty} \delta(t)\, dt \underbrace{=}_{\text{by defn.}} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} g_\epsilon(t)\, dt = \lim_{\epsilon \to 0} 1 = 1.$$

Note that the limit of the integrals of g_n does converge! We can derive a few other properties as well. The key property is that

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0) \quad (3)$$

[1]This treatment is quite informal; the rigorous/correct way to define it is beyond the scope of the course and not needed. One either needs the theory of distributions or measures for a proper treatment.

for any continuous function f. The idea here is that the δ is zero away from $t = 0$, so

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = \int_{-\infty}^{\infty} \delta(t) f(0)\, dt = f(0) \int_{-\infty}^{\infty} \delta(t)\, dt = f(0).$$

One can more rigorously show the property with the g_ϵ's and a limit. In 'function' form, the identity is sometimes written as

$$f(t)\delta(t) = f(0)\delta(t).$$

In fact, δ can be manipulated in expressions as if it were a function **most of the time** (but not always; we will not run into any trouble here, however!). Allowed operations include scaling by a constant and translation. The expression

$$a\delta(t - t_0)$$

has an integral of a (so an impulse/mass of a), concentrated at the point t_0. Translating the integral in (3), we find that

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0).$$

The $\delta(t - t_0)$ selects the value of f at the point where its mass is concentrated. The main properties are summarized below, along with a diagram:

Properties of the Dirac delta: The Dirac delta $\delta(t)$ has the following properties:

$$(i) \qquad \int_a^b \delta(t)\, dt = 1 \text{ if } [a,b] \text{ contains } 0$$

$$(ii) \qquad \int_a^b \delta(t)\, dt = 0 \text{ if } [a,b] \text{ does not contain } 0$$

$$(iii) \qquad \int_{-\infty}^{\infty} \delta(t - a) f(t)\, dt = f(a) \quad \text{for all continuous functions } f(t).$$

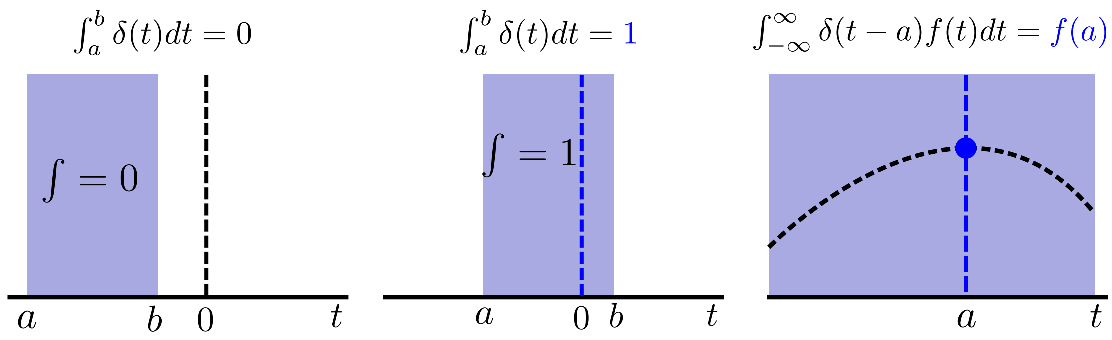
In 'function' notation, the properties can be equivalently written as

$$(i') : \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$(ii') \qquad \delta(t) = 0 \text{ if } t \neq 0$$

$$(iii') \qquad f(t)\delta(t - a) = f(a)\delta(t - a).$$

Note that (ii') and (iii') are just shorthand; they are 'practical' rules for calculation only.



(i) (ii) (iii)

Technical note: I am deliberately ignoring an unpleasant technicality when one of the interval endpoints is zero exactly. The issue is that the g_ϵ's are partly defined on $t < 0$, which is not compatible with the $t \geq 0$ domain of the Laplace integral. However, as the technicality will not come up, it will not be addressed further.

3 Laplace transform

By using the rules, it is easy to compute the Laplace transform. Using the 'function version', we can compute

$$\begin{aligned}
\mathcal{L}[\delta(t-a)] &= \int_0^\infty e^{-st}\delta(t-a)\,dt \\
&= \int_0^\infty e^{-as}\delta(t-a)\,dt \quad \text{by (iii')} \\
&= e^{-as}\int_0^\infty \delta(t-a)\,dt \\
&= e^{-as} \quad \text{by (i') and (ii'), since } a > 0
\end{aligned}$$

Using the integral version,

$$\begin{aligned}
\mathcal{L}[\delta(t-a)] &= \int_0^\infty e^{-st}\delta(t-a)\,dt \\
&= e^{-st}\Big|_{t=a} \quad \text{by (iii)} \\
&= e^{-as}.
\end{aligned}$$

If $a = 0$ then we get the important formula

$$\mathcal{L}[\delta(t)] = 1.$$

At last, we can inverse transform constants! Note that while δ is not a function, its Laplace transform **is a function**. Like step functions, δ's are easy to deal with once transformed.

Moreover, the convolution theorem tells us that

$$\mathcal{L}[f * \delta] = \mathcal{L}[f]\mathcal{L}[\delta] = \mathcal{L}[f]$$

and so

$$f * \delta = f.$$

Thus δ is the 'identity function' for convolutions. This makes sense since we have shown that δ is the multiplicative identity in the transform space so it should be the convolution identity in regular space.

Further properties of the delta: The Dirac delta δ has the Laplace transform

$$\mathcal{L}[\delta(t)] = 1$$

and if $a > 0$ then

$$\mathcal{L}[\delta(t-a)] = e^{-as}.$$

Furthermore, if f is a (piecewise) continuous function then

$$f * \delta = f.$$

4 Use in solving DEs

Now it is simple to solve ODEs by the standard method. As an example, consider a pendulum with angular displacement $y(t)$ is governed (approximately) by the ODE

$$y'' + y = g(t)$$

where g is the applied force. If it is let go at $y(0) = 1$ then the pendulum will swing back and forth from -1 to 1 with period 2π. Suppose we push the pendulum after it swings back to $y(0) = 1$ at $t = 2\pi$ with an impulse of 1, then again at $t = 4\pi$ with the same impulse. Then $y(t)$ is the solution to

$$y'' + y = \delta(t-\pi) + \delta(t-2\pi), \qquad y(0) = 1, y'(0) = 0.$$

Take the Laplace transform of the IVP:

$$s^2 Y + s = e^{-2\pi s} + e^{-4\pi s}.$$

This gives

$$Y = \underbrace{-\frac{s}{s^2+1}}_{} Y_h + \underbrace{H(s)(e^{-2\pi s} + e^{-4\pi s})}_{} Y_p, \quad H(s) = \frac{1}{s^2+1}.$$

For the first term:

$$y_h = \mathcal{L}^{-1}(Y_h) = -\cos t.$$

For the second part, Y_p, we recognize the e^{-cs} terms and apply the step function rule to $e^{-2\pi s}H(s)$ and $e^{-4\pi s}H(s)$ to obtain

$$y(t) = -\cos t + h(t-2\pi)u_{2\pi}(t) + h(t-4\pi)u_{4\pi}(t)$$

where $h = \mathcal{L}^{-1}(H)$. Now since

$$h = cL^{-1}(1/(s^2+1)) = \sin t$$

we get the solution

$$y(t) = -\cos t + \sin(t-2\pi)u_{2\pi}(t) + \sin(t-4\pi)u_{4\pi}(t).$$

It happens to be the case that $\sin(t-2\pi)$ and $\sin(t-4\pi)$ both equal $\sin t$ so

$$y(t) = -\cos t + \sin(t)(u_{2\pi}(t) + u_{4\pi}(t)).$$

It is revealing to write this out case by case. We have, as a piecewise function,

$$y(t) = -\cos t + \begin{cases} 0 & t < 2\pi \\ \sin t & 2\pi < t < 4\pi \\ 2\sin t & t > 4\pi \end{cases}.$$

The first push causes an oscillation $\sin t$ (of the same frequency as the natural one). The second push adds another one, and the amplitude is now doubled. We are pushing the pendulum in sync with its oscillation, causing its amplitude to grow. See the homework for what happens with further pushes.

Another example: We solve

$$y'' + y' + 2y = \delta(t-1), \quad y(0) = 1, y'(0) = 0.$$

Applying the Laplace transform, we find that

$$(s^2+s+2)Y - s = e^{-s},$$

and so

$$Y(s) = \frac{s}{(s-1)(s+2)} + e^{-s}H(s), \qquad H(s) = \frac{1}{(s-1)(s+2)}.$$

The first term is the homogeneous part; the second is the response to the δ forcing. We then apply the rule $\mathcal{L}[f(t-c)u_c(t)] = e^{-cs}F(s)$ to obtain the inverse transform:

$$y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t} + h(t-1)u_1(t)$$

where $h = \mathcal{L}^{-1}[H] = \frac{1}{3}e^{-t} - \frac{1}{3}e^{2t}$, which is obtained from partial fractions:

$$H(s) = \frac{1/3}{s-1} - \frac{1/3}{s+2}.$$

5 Impulse response

Recall that the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y'(0) = 0$$

has a transformed solution

$$Y(s) = H(s)G(s)$$

where $H(s) = 1/(as^2 + bs + c)$ is the transfer function. We used this and the convolution theorem to then show that the solution to the IVP is

$$y(t) = (h * g)(t)$$

where $h(t) = \mathcal{L}^{-1}[H(s)]$.

Now observe that when $g = \delta(t)$, the transform/solution are simply

$$Y(s) = H(s), \quad y(t) = h(t).$$

For this reason, the function $h(t)$ is called the **impulse response** of the system: the response to a unit impulse with zero initial conditions, i.e. the solution to

$$a'' + by' + cy = \delta(t), \quad y(0) = y'(0) = 0.$$

More generally, we have the following important principle: For a system governed by a linear constant coefficient ODE, **the response of the system to an input $g(t)$ is the convolution of the impulse response with the input.**

5.1 Aside: superposition of deltas

Imagine a pendulum with an applied force, with a model like

$$y'' + y = g(t).$$

Now think of $g(t)$ as a series of small pushes:

$$g(t) = \sum_{k=1}^{\infty} \Delta t g(x_k)\delta(t - x_k)$$

where $x_k = k\Delta t$ and each has a magnitude of $\Delta t g(x_k)$. The response is then

$$y(t) = \sum_{k=1}^{\infty} \Delta t g(x_k)h(t - x_k)$$

which is a superposition of impulse responses to impulses at each x_k.

This provides an interpretation for the solution for the actual $g(t)$,

$$y(t) = \int_0^t h(t-s)g(s)\,ds.$$

The solution is a superposition (by way of an integral rather than a sum) of responses to impulses at each point s:

$$\text{impulse at } s\text{: } g(s)\Delta s \to \text{response: } h(t-s)g(s)\,ds$$

6 Back to resonance...

Let's look again at the resonance example, where an oscillator is driven by a forcing at a frequency ω:

$$y'' + a^2 y = f(t), \qquad f(t) := \sin \omega t.$$

Assume that $y(0) = y'(0) = 0$ for simplicity.

Taking the Laplace transform, we get

$$Y(s) = \frac{1}{(s^2 + a^2)(s^2 + \omega^2)}.$$

To invert, we can use partial fractions. There are two **distinct** quadratic factors (or four linear factors using the complex version) and so

$$Y(s) = \frac{c_1 s + c_2}{s^2 + a^2} + \frac{c_3 s + c_4}{s^2 + \omega^2}$$

which correspond to the four terms

$$\sin(at), \quad \cos(at), \quad \sin(\omega t), \quad \cos(\omega t).$$

However, if $\omega = a$ then

$$Y(s) = \frac{1}{(s^2 + a^2)^2}.$$

The previous form no longer applies. That makes sense, because this is the resonant case - so we expect a qualitatively different result.

The inverse can be computed using the convolution theorem ($\sin(at) * \sin(at)$) or using partial fractions in the complex form:

$$Y(s) = \frac{1}{(s - ia)^2 (s + ia)^2} = \cdots$$

The point here is that we see terms like

$$\frac{\cdots}{(s - ia)^2} \to te^{ait} \implies t\cos(at), \quad t\sin(at)$$

in the solution, i.e. resonance. Compare that to the non-resonant case where we instead get

$$\frac{1}{(s - ia)(s - i\omega)} \to c_1 e^{iat} + c_2 e^{i\omega t}$$

leading to regular oscillating terms. Thus, a pole of 'order' more than one (i.e. a root in the denominator is repeated), corresponds to the resonant case. In particular, this means we can 'spot' resonance in the oscillating system by looking in the Laplace domain!

The 'natural' frequencies appear as poles of the transfer function (e.g. $\pm ai$ here) in the complex plane. An input forcing will resonate if its Laplace transform also has a pole that coincides with the natural ones.