Math 353 Lecture Notes
Second Order Linear ODEs: inhomogeneous problems

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Topics covered

- Second order, linear ODEs: inhomogeneous problems
  - Homogeneous plus particular solutions
  - Undetermined coefficients (for LCC ODEs)
  - Variation of parameters
- A few more notes:
  - Example: resonance (forced oscillations)
  - Reduction of order

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1 inhomogeneous linear ODEs

Consider the general linear second order ODE

$$y'' + p(t)y' + q(t)y = f(t) \tag{I}$$

letting L be defined in the usual way.

Suppose we have a **particular solution** y_p to (I). Then

$$L[y - y_p] = Ly - Ly_p = 0$$

so $y - y_p$ is a solution to the homogeneous problem. Thus, to solve the **inhomogeneous** problem, it suffices to:

- find one solution y_p (by some means)
- Solve the homogeneous problem to get $y_h = \cdots$ (general solution
- Combine to get $y = y_h + y_p$

Thus, we only need to develop a way to find a particular solution.

1.1 undetermined coefficients

The first approach is valid only for LCC equations

$$y'' + by' + cy = f(t).$$

The operator $Ly = y'' + by' + cy$ sends functions to functions:

$$v \to Lv$$

and the solution y is the function such that L sends it to f.

To solve for y, we can try to go 'in reverse' as follows:

- Plug some typical functions into L to see the pattern of $v \to Lv$'s
- Look for known Lv's in f
- 'Invert' using the known pattern (from Lv back to v) to get y

From inspsection calculation, some common patterns are as follows. Here P_k and Q_k denote polynomials of degree at most k, e.g. P_2 denotes $c_2t^2 + c_1t + c_0$.

$$\begin{aligned} \text{exponentials:} \quad & L[e^{\lambda t}] = p(\lambda)e^{\lambda t} \\ \text{sines/cosines:} \quad & L[a \sin t + b \cos t] = c \sin t + d \cos t \\ \text{polynomials:} \quad & L[P_k(t)] = Q_k(t) \end{aligned}$$

These can be combined; more generally, we have

$$L[P_k \cdot e^{\lambda t}] = Q_k \cdot e^{\lambda t}$$

and most generally,

$$(\text{poly. of deg. } k) \cdot (\text{sines and cosines}) \cdot e^{\lambda t}$$

is sent by L to an expression of the same form.

Shortcut: It follows from the above rule that if the expression (**??**) shows up on the right hand side, you construct a guess by replacing sin/cos with a linear combination of sin and cos and any polynomial with an arbitrary one of the same degree (e.g. $t \to (at + b)$ and $\cos t \to u \cos t + v \sin t$).

Often, such a guess is overkill, but it does provide enough generality to ensure you have a solution of the right form.

Here's a straightforward example. Consider

$$y'' + y' + y = \sin 2t.$$

Guess $y = a \sin 2t + b \cos 2t$ and plug in:

$$(-4a - 2b + a) \sin 2t + (-4b + 2a + b) \cos 2t = \sin 2t$$

so $-3a - 2b =$ and $-3b + 2a = 0$ which gives $b = -2/13$ and $a = -3/13$. Thus

$$y_p = -\frac{3}{13} \sin 2t - \frac{2}{13} \cos 2t$$

is a particular solution.

Another example: Suppose we want to solve

$$y'' + y' + y = t^2 e^{3t} + e^{2t} \sin t.$$

By linearity (superposition), we can find particular solutions for each term separately.

Following the rules, we need to guess

$$y_{p1} = (at^2 + bt + c)e^{3t}, \qquad y_{p2} = e^{2t}(p \sin t + q \cos t)$$

You can then plug both into the ODE and solve for coefficients to get

$$y_{p1} = \left(\frac{1}{13} t^2 - \frac{14}{169} t + \frac{72}{2197} \right) e^{3t}, \quad y_{p2} = e^{2t} \left(\frac{6}{61} \sin t - \frac{5}{61} \cos t \right)$$

Note the calculations are rather tedious, so for most problems, a computer algebra package is best (I used Wolfram Alpha here). However, sometimes there is some nice symmetry or other insight that helps one to solve without plugging into a computer.

1.2 Undetermined coefficients: important exception

However, there is one crucial exception (this is really the important part of the discussion - the rest is just computation).

Suppose the right hand side is **a solution to the homogeneous problem**, such as

$$y'' - 9y = 5e^{3t}.$$

The rule would say to guess Ce^{3t}, but L applied to this just gives zero!

Thus, it cannot give a non-zero right hand side.

The fix is to **multiply by t until it is not a homogeneous solution**. That is, we keep multiplying the 'base' guess from the procedure by t until it stops vanishing.

Why does this work? Essentially, one has to show that if $e^{\lambda t}$ is a solution and λ is a repeated root then

$$L[P_k e^{\lambda t}] = Q_{k-1} e^{\lambda t}$$

i.e. the degree of the polynomial gets reduced by one. if instead λ is a root that appears s times then the degree is reduced by s instead.

It follows that the modified rule should be:

- Construct the base guess according to the previous rules
- Multiply it by t^s where s is the multiplicity of λ (the number of times it appears)

Example 1: Suppose
$$y'' - y = e^t.$$
Since $\lambda = 1$ is a root of the char. polynomial (e^t is a homogeneous solution) we must choose
$$y_p = ate^t$$
i.e. multiply the usual guess ae^t by t.

Example 2: For the equation
$$y'' - 2y' + y = t^2e^t$$
the usual guess would be
$$(at^2 + bt + c)e^t.$$
However, $\lambda = 1$ is a double root, so we must multiply by t^2:
$$y_p = {\color{red}t^2}(at^2 + bt + c)e^t.$$
Note that this ensures that no term of y_p vanishes when L is applied - a te^t or e^t would do nothing, so it makes sense the lowest-degree term is t^2e^t.

1.3 Variation of parameters

A method for obtaining a particular solution that works for **any** second order, linear ODE

$$y'' + p(t)y' + q(t)y = f(t).$$

It requires knowing a solution basis $\{y_1, y_2\}$ for the homogeneous problem, which mays not be available for every ODE. It does, however, show that the particular part is easy to find once the homogeneous part is solved.

1.4 Starting point: for first order ODEs

To ge the idea, let's go back to the first order linear problem

$$y' + p(t)y = f(t).$$

This was solved explicitly with an integrating factor.

There is another (equivalent) way. Suppose we have a homogeneous solution $y_h(t)$. The idea of **variation of parameters** is to look for a solution

$$y_p(t) = v(t)y_h(t)$$

i.e. a 'multiple' of the homogeneous solution, but with varying coefficients.

Plugging this into the ODE, we get

$$v'y_h + vy_h' + pvy_h = f$$

but the last two terms vanish since $y_h' + py_h = 0$ so this simplifies greatly:

$$v'y_h = f \implies v = \int \frac{1}{y_h} f\, dt$$

It follows that a particular solution is

$$v = y_h \int \frac{1}{y_h} f\, dt$$

which matches the solution we got with integrating factors (check this!).

1.5 Second order ODEs: derivation (optional)

The same trick works for second order ODEs as well. Consider

$$y'' + p(t)y' + q(t)y = f(t).$$

Suppose we have linearly independent solutions y_1 and y_2 to the **homogeneous** problem.

The result is best derived by using system notation - so that the first order system is solved similar to the example of the previous section for first order ODEs. Let

$$\Phi(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_2'(t) & y_2'(t) \end{bmatrix}$$

which is called the **fundamental matrix** (which we encountered earlier!). Further, let

$$\mathbf{x}(t) = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}, \qquad A(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$$

so that the ODE reads

$$\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{f}$$

It follows that (looking at each column separately)

$$\Phi'(t) = A(t)\Phi(t).$$

Now we look for a solution

$$\mathbf{x}_p = \Phi(t)\mathbf{v}(t)$$

and plug in to find that

$$\Phi'(t)\mathbf{v}(t) + \Phi(t)\mathbf{v}'(t) = A(t)\Phi(t)\mathbf{v}(t) + bff.$$

Two of the terms cancel after plugging in $\Phi' = A(t)\Phi(t)$ (here we are using the fact that Φ is a solution to the homogeneous system), leaving

$$\Phi(t)\mathbf{v}'(t) = \mathbf{f} \implies \mathbf{v} = \int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds$$

where the lower integration limit can be anything.

Now we just have to unpack this and convert back. With $\mathbf{v} = (v_1, v_2)$, we have

$$\mathbf{x}_p = \Phi(t)\mathbf{v}(t) \implies \begin{cases} y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \\ y'(t) = v_1(t)y_1'(t) + v_2(t)y_2'(t) \end{cases}$$

and the formula then reads

$$\begin{bmatrix} y_p(t) \\ y_p'(t) \end{bmatrix} = \Phi(t) \int^t \Phi^{-1}(s) \begin{bmatrix} f(s) \\ 0 \end{bmatrix} ds.$$

1.6 The result

Finally, if you want a formula for $y(t)$ only, take the first row. Note that

$$\Phi^{-1} = \frac{1}{W(t)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$$

where $W(t) = \det(\Phi)$ is the Wronskian. Then

$$y_p = \underbrace{\left(-\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(s)}\,ds\right)}_{v_1} y_1 + \underbrace{\left(\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)}\,ds\right)}_{v_2} y_2 \tag{V}$$

The value of t_0 a free choice.

The point (how to use VoP): Given the **homogeneous solutions** to the second-order linear ODE $Ly = 0$, one can solve the inhomogeneous problem $Ly = f$ for any right hand side.

In this sense, all the interesting structure is in the *homogeneous* part, and the inhomogeneous part we get for free.

The computation can be done just by 'plugging in' to the VoP formula. However, it can be messy (the integrals are not nice), so an inspired guess with undetermined coefficients can be faster.

Example 1: Suppose we seek a solution to

$$t^2 y'' - 2y = 3t^2 - 1, \qquad t > 0,$$

given linearly independent solutions $y_1 = t^2$ and $y_2 = 1/t$ (from earlier). First, compute

$$W(y_1, y_2) = -3.$$

Note that because L in the formula is assumed to be $y'' + \cdots$ we need to divide the equation by t^2:

$$y'' - 2y = 3 - 1/t^2.$$

Substituting everything into the formula, we get

$$y_p = \frac{t^2}{3} \int^t \frac{1}{s}(3 - \frac{1}{s^2})\,ds - \frac{1}{3t} \int^t s^2(3 - \frac{1}{s^2})\,ds.$$

This evaluates to

$$y_p = t^2 \ln t + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3}.$$

Note that the t^2 term can be dropped from y_p. The general solution is then

$$y = c_1 t^2 + \frac{c_2}{t} + t^2 \ln t + \frac{1}{2}.$$

Example 2: Consider
$$y'' + y = g(t)$$
for an *arbitrary* $g(t)$. Choose $y_1 = \cos t$ and $y_2 = \sin t$ so that $W(y_1, y_2) = 1$. We get

$$y_p = -\cos t \int_0^t \sin(s) g(s)\, ds + \sin t \int_0^t \cos(s) g(s)\, ds.$$

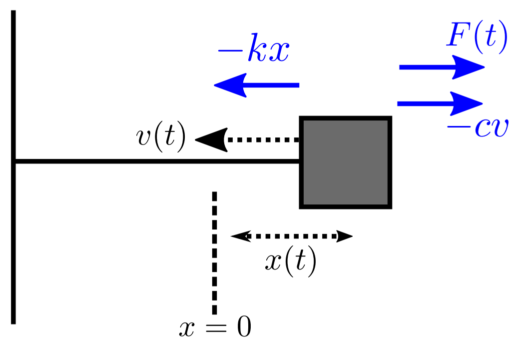
Putting this into one integral and using a trig. identity:

$$y_p = \int_0^t \sin(t - s) g(s)\, ds.$$

We'll revisit this later in studying the Laplace transform.

2 Application: oscillators and resonance

Consider a mass on a spring with displacement $x(t)$ (unstretched position: $x = 0$) and velocity $v(t) = dx/dt$. The spring has a restoring force $-kx$ and there is resistance proportional to velocity (e.g. friction). There is an external force $F(t)$. Forces are shown in in blue below.



Newton's third law says that

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t).$$

There are too many parameters here to consider. We can easily simplify by **scaling**. Define a scaled time $s = t/T$, where T is a constant (in units of time) and write $' = d/ds$ to get

$$x'' + \frac{Tc}{m}x' + \frac{kT^2}{m}x = \frac{T^2}{m}F(t)$$

since
$$\frac{dx}{dt} = \frac{1}{T}\frac{dx}{ds}$$
Choose the timescale $T = \sqrt{m/k}$ and set $\beta = Tc/m$. Then

$$x'' + \beta x' + x = f(s)$$

where $f(s)$ is the scaled forcing as a function of τ (what, exactly, is f in terms of F?).

Thus, to understand the dynamics, it is enough to study the equation

$$x'' + \beta x' + x = f(s). \tag{1}$$

That is, the qualitative behavior really depends only on the parameter $\beta = c/\sqrt{mk}$.

2.1 Application: resonance

Now suppose there is no damping $\beta = 0$. (Note that t is used here in place of τ in (1) for convenience). If there is also no external force ($f = 0$) then

$$y = c_1 \cos t + c_2 \sin t.$$

The spring oscillates with a period of 2π (its 'natural' period) in non-dimensional time.[1] Now suppose that there is some additional forcing:

$$x'' + x = A \sin \omega t.$$

First case: If $\omega \neq 1$, we can solve this using undetermined coefficients by guessing

$$x_p = a \sin \omega t + b \cos \omega t.$$

Plugging in, we get

$$a(1 - \omega^2) \sin t + b(1 - \omega^2) \cos t = A \sin \omega t$$

so $b = 0$ and $a = A/(1 - \omega^2)$. A particular solution is then

$$x_p = \frac{A}{1 - \omega^2} \sin \omega t.$$

Thus, the forcing causes the spring to oscillate with the same frequency as the input. The general solution is then

$$x(t) = c_1 \cos t + c_2 \sin t + x_p(t).$$

The solution is a superposition of two oscillations: one at the **natural frequency** 1 and one at the **forcing frequency** ω.

Second case: If $\omega = 1$ then the forcing is a homogeneous solution. Thus, undetermined coefficients says the particular solution has an extra factor of t:

$$x_p(t) = t(a \sin t + b \cos t).$$

Note that, after a calculation,

$$\begin{aligned} x_p''(t) &= 2a \cos t - at \sin t - 2b \sin t - bt \cos t \\ &= 2a \cos t - 2b \sin t - x_p(t). \end{aligned}$$

[1] In dimensional terms, $2\pi T = 2\pi\sqrt{k/m}$ and its reciprocal is the **natural frequency** $\nu = \frac{1}{2\pi}\sqrt{m/k}$.

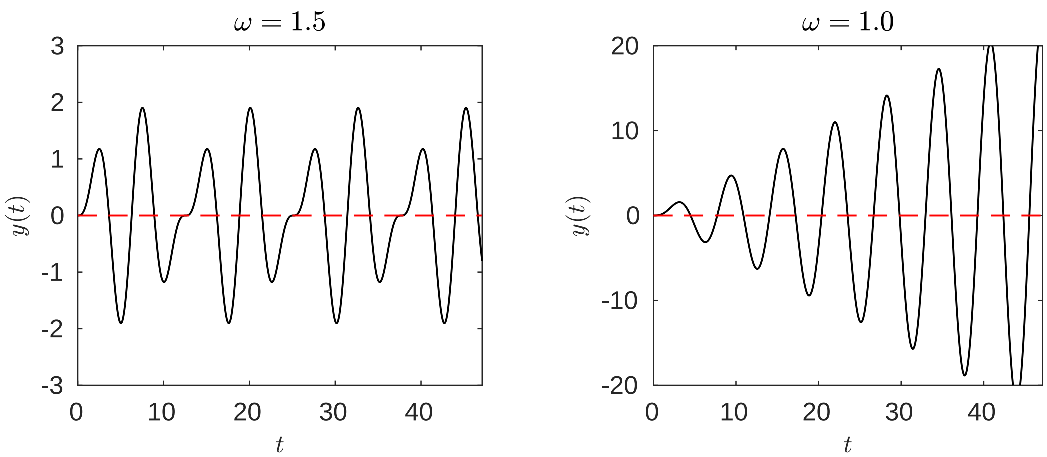
Plugging in x_p into the ODE therefore has most terms cancel, yielding

$$x_p''(t) + x_p(t) = A \sin t$$
$$\implies 2a \cos t - 2b \sin t = A \sin t$$
$$\implies a = 0 \text{ and } b = -A/2.$$

It follows that a particular solution is

$$x_p(t) = -\frac{A}{2} t \cos t.$$

Thus the displacement of the spring *increases linearly in magnitude* and oscillates.



Interpretation: This phenomenon is called **resonance**: a system that normally has bounded solutions can be forced at its natural frequency to cause the amplitude to grow over time.

You know resonance from physics - for instance, pushing a swing at just the right times to gain height, or exciting a vibrating system by driving it at a natural frequency.

Our analysis shows that the solution **stays bounded** (i.e. the amplitude has a maximum value) unless the forcing matches the natural frequency.

Only if it matches that frequency will there be a growth in the amplitude.

We can think of the system as being 'stable' if all solutions stay bounded (so the oscillations never get out of control). We see that without damping, the system is **just barely stable** in the sense that a small forcing of the right frequency makes solutions unbounded, e.g.

$$y'' + y = 0.0001 \sin t$$

will still have solutions of growing amplitudes. On the other hand, such systems are rare in physical reality, since there is usually some energy loss (damping, friction etc.), especially when the motion becomes more dramatic!

3 Stray notes: reduction of order

There is no general way to find the homogeneous solutions to a second order ODE

$$y'' + p(t)y' + q(t)y = 0.$$

However, if we know **one** solution y_1, it is possible to get the other one.

The trick here is, essentially, variation of parameters. Guess

$$y_2 = v(t)y_1$$

where $v(t)$ is like a time-varying coefficient. Plugging into the ODE yields a new equation for v. After some simplification, we discover that the ODE for v is actually solvable exactly.

In detail, plug into the ODE:

$$(vy_1)'' + p(vy_1)' + qy = 0$$

$$\implies v''y_1 + 2v'y_1' + vy_1'' + pvy_1' + pv'y_1 + qvy_1.$$

Now group all the v' and v terms together:

$$y_1v'' + (2y_1' + py_1)\, v' + v\,(y_1'' + py_1' + qy_1) = 0.$$

But the last term on the right is zero since y_1 is a solution! We are left with something like

$$v'' + g(t)v' = 0.$$

While technically a second order ODE for v, it's a **first order ODE for** v'. Moreover, it is one of the exactly-solvable types. Thus we can solve for v', then integrate to get v.

Example: repeated roots case

The method applies nicely to the repeated roots case for LCC ODEs. Consider

$$y'' - 2by' + b^2y = 0.$$

which has a solution $y_1(t) = e^{bt}$ (but no other exponential solutions.

Reduction of order can be used here, guessing a solution of the form

$$y_2 = v(t)e^{bt}$$

Plugging in this guess for y_2 into the ODE gives

$$(b^2v + 2bv' + v'')e^{bt} - 2b(bv + v')e^{bt} + b^2ve^{bt} = 0.$$

You can check that almost all terms cancel here, leaving

$$v'' = 0 \implies v(t) = c_1t + c_2$$

so te^{bt} is also a solution (yielding the desired basis $\{e^{bt}, te^{bt}\}$).