Math 353 Lecture Notes
Second Order Linear ODEs: fundamentals

Fall 2020

Topics covered

- Fundamentals
  - Review of vector spaces
  - Homogeneous 2nd order ODEs: structure
  - Connection to linear systems in $\mathbb{R}^2$
  - Linear independence of solutions and functions
  - Testing linear independence (Wronskian)
- Solution procedures
  - Linear constant coefficient (LCC) ODEs
  - Detail: choosing a nice basis
  - Extension to Higher-order LCC ODEs
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1 Interlude: linear algebra review

1.1 A quick summary

To setup, let's review the analogous theory you know from linear algebra. The example for vectors in $\mathbb{R}^2$ is listed in the next section, but is meant to be in parallel to this one.

A **(real) vector space** V is a set of 'vectors' v with the property that linear combinations stay in V, i.e.

$$v_1, v_2 \in V,\ c_1, c_2 \text{ scalars } \implies c_1 v_1 + c_2 v_2 \in V$$

where a 'scalar' is a real number.[1]
The standard example is $\mathbb{R}^n$, the space of n-dimensional vectors $(x_1, \cdots, x_n)$.

[1] A 'complex' vector space uses comple x numbers for scalars, which may be useful later.

The **span** of some vectors is the set of all linear combinations:

$$\text{span}(x_1, \cdots, x_k) = \{c_1 x_1 + \cdots + c_k x_k \text{ for scalars } c_1, \cdots, c_k\}.$$

A set of vectors is **linearly independent** if none of the vectors are in the span of the others. This means, critically, that there are a 'minimal' set that spans in that we can't remove any without reducing the span. Precisely,

$$x_1, \cdots, x_k \text{ are linearly independent if and only if...}$$
$$c_1 x_1 + \cdots + c_k x_k = 0 \text{ implies } c_j = 0 \text{ for all } j.$$

To rephrase the last line: there is no non-trivial way to make zero as a linear combination of the set of vectors $x_1, \cdots x_k$.
Or, again: there is no way to write any of the x_j's as a linear combination of the others.

A **basis** for a vector space V is a linearly independent set of vectors that spans the space. Linear independence ensures all the vectors in the basis are needed.
The **dimension** of V is the size of a basis.

Critically, this means that we can represent any vector in V as a linear combination of the basis vectors.

A **linear operator** on a vector space is a function L that takes vectors to vectors and is **linear**, which means that

$$L[c_1 v_1 + c_2 v_2] = c_1 L[v_1] + c_2 L[v_2], \qquad \text{for scalars } c_1, c_2 \text{ and vectors } v_1, v_2.$$

Here $L[v]$ denotes L applied to v (you usually just see Lv for linear operators).

You can break this up into 'scaling' and 'additivity' properties if you like:

$$L[cv] = cL[v], \quad L[v + w] = L[v] + L[w].$$

Why is this useful? Suppose we want to understand a vector space V and an operator L. A strategy is to first find a basis $\phi_1, \cdots, \phi_n$. Then

$$v = c_1 \phi_1 + \cdots + c_n \phi_n$$

for any vector v. Applying L and **using linearity**, we get

$$L[v] = c_1 L[\phi_1] + \cdots + c_n L[\phi_n].$$

Thus, to understand the function $L[v]$, we only need to know

- A basis $\phi_1, \cdots, \phi_n$
- what happens when L is applied to each basis element

That reduces 'know what L does on the n-dimensional V' to 'know what L does on n things'.

Similarly, if we want to understand some vector space V, it is enough to find a basis and then look at all linear combinations of that basis. That is, we need only get n linearly independent elements and the rest is easy.

1.2 Example: vectors in $\mathbb{R}^2$

Here we review the concepts above for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Linear independence: In $\mathbb{R}^2$, two vectors v, w are linearly independent if they are not in the same direction, i.e. if

$$v \neq cw \text{ for any } c.$$

An example of linearly independent vectors:

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Recall: **a matrix is invertible if and only if its columns are linearly independent.**

This means that to check linear independence, we can look at the matrix whose columns are those vectors:

$$v_1, v_2 \text{ linearly independent } \iff [v_1|v_2] \text{ is invertible.}$$

For example, with v_1, v_2 above,

$$[v_1 \,|\, v_2] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \qquad \det \;= 3 - 2 = 1 \neq 0$$

which verifies linear independence. (We'll use this shortly for DEs).

Note that in $\mathbb{R}^3$, the 'multiples of each other' rule is not enough, since one vector could be a linear combination of the other two, e.g. the set

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

is not linearly independent because $v_3 = 2v_1 + 2v_2$.

Basis: In $\mathbb{R}^2$, the above means any two vectors that are not multiples form a basis... e.g.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

as before. Any vector (x, y) can be written in terms of the basis as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

As noted before, the matrix whose columns are v_1, v_2 is invertible, which makes sense since

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

has a solution for **any** (x, y) only if the matrix can be inverted.

Linear operators: In R^n, linear operators are the functions

$$L[v] = Av, \qquad A = n \times n \text{ matrix.}$$

(Technically, an operator can send v to a different space, where A could be $m \times n$ instead, but that's not needed here).

Linearity is easily checked directly from the definition of matrix multiplication.

1.3 More on linear algebra in R^n

Eigenvalues: An essential concept (which we'll use extensively in the course) is the **eigenvalue/eigenvector** pair.

Given a matrix A, we say v is an **eigenvector** and λ is an eigenvalue if

$$Av = \lambda v.$$

Geomtrically, this says:

- A applied to v stays in the direction of v
- A scales the vector v by λ.

Both properties are useful in theory as welll. Suppose we have an $n \times n$ matrix A with n eigenvectors $v_1, \cdots v_n$ that are **linearly independent** and eigenvalues $\lambda_!, \cdots \lambda_n$.

This is a **basis of eigenvectors** for $\mathbb{R}^n$. That means that any vector x can be written

$$x = c_1 v_1 + \cdots c_n v_n.$$

Now, applying A to this vector, we see that it leaves each term in the same direction ($c_1 v_1$ still in the v_1 direction, etc.) and

$$Ax = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n.$$

Thus, the application of A to x can be described as

scale the j-th component of x (in the eigenvector basis) by λ_j.

This reduces Ax to n simple operations. While useful, of course, in R^n, we will find the analogous notion for differential equations to provide even more significant benefits.

One last remark: An important fact is that

eigenvectors of distinct eigenvalues are linearly independent.

In particular, if A is an $n \times n$ matrix and it has n **distinct eigenvalues**, then its eigenvectors are autonomatically a basis for R^n.;.

Linear systems and null spaces: For a linear system

$$Ax = b$$

we know that a unique solution exists if and only if A is invertible.

If A is **not** invertible, then it has a non-trivial **null space**

$$N = \{x : Ax = 0\}.$$

In this case, any 'particular' solution x_0 to $Ax = b$ cannot be unique, since

$$Ax_0 = b \implies x = x_0 + y \text{ solves } Ax = b \text{ for any } y \in N.$$

That is, we can add any solution to $Ay = 0$ to our particular solution. For instance, take

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

A particular solution and the null space are

$$x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{null space } = \{y : \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} y = 0\} = \text{span}(\begin{bmatrix} 2 \\ -1 \end{bmatrix})$$

The set of solutions is then

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

2 First order linear ODEs (in terms of vector spaces)

To gain some intuition, let's look at **linear** first order ODEs

$$y' + p(t)y = g(t).$$

We will need the definitions that the ODE is...

- **homogeneous** if $g(t) = 0$
- **inhomogeneous** If $g(t)$ is non-zero

This equation has the form

$$L[y] = q, \qquad L[y] := y' + p(t)y.$$

We happen to have the full solution. Let $\phi(t)$ the the integrating factor (what is it?). Then

$$y(t) = \frac{C}{\phi(t)} + \frac{1}{\phi(t)} \int_a^t \phi(s)g(s)\, d\, ds$$

The solution can be written as

$$y(t) = Cy_1(t) + y_p(t)$$

where

$$y_1(t) = \frac{1}{\phi(t)}, \quad y_p = \frac{1}{\phi(t)} \int_{t_0}^{t} \phi(s)g(s)\, ds.$$

Now observe that y_1 and y_p solve

$$L[y_1] = 0, \quad L[y_p] = g.$$

The associated **homogeneous** problem for this operator is

$$L[y] = 0.$$

We see, then, that the general form for solutions to

$$y' + p(t)y = g(t)$$

has the form

$$y = \text{homogeneous solution} + \text{ particular solution}$$

where the **particular solution** is **one** solution to the original problem and the **homogeneous solution** is the general solution to the homogeneous problem

$$y' + p(t)y = 0.$$

Notably, the set of homogeneous solutions is a vector space (with dimension 1):

$$V = \{y : L[y] = 0\} = \{cy_1 : c \in \mathbb{R}\} = \text{span}(\{y_1\}). \tag{1}$$

This vector space structure will be crucial once the dimension becomes bigger than one.

Why this structure? For the non-homogeneous problem,

$$L[y] = g(t), \tag{2}$$

we have a solution y_p (with no free constant!). How does this connect to the homogeneous case? The idea is that the single function y_p 'takes care of' the right hand side, and the rest of the solution is just the homogeneous part (which we just solved for).

Formally, observe that if y is any other solution to (2) then

$$L[y - y_p] = L[y] - L[y_p] = g - g = 0.$$

Thus $y - y_p \in V$ (the set of homogeneous solutions (1)), which means that

$$y = cy_1 + y_p.$$

This is the general solution to (2). Note that the trick of subtracting off a particular solution to get rid of g really only required that L was linear (not important that it was also first order). The principle applies quite generally and will be used often in our study of DEs.

3 Second order linear ODEs: context

3.1 A first example

Before getting to the general theory, let's explore the structure with an example. Consider the **second order** linear ODE (for $y(t)$)

$$y'' + y' - 2y = 0$$

Note that the operator here is $Ly = y'' + y' - 2y$, and the ODE is $Ly = 0$.

Let's search for solutions by the method of guessing. We know that e^{rt} is a simple solution, so guess a solution of the form

$$y(t) = e^{rt}.$$

Plugging this into the ODE we find that

$$e^{rt} \text{ is a soln.} \iff r^2 e^{rt} + re^{rt} - 2e^{rt} = 0$$

which then simplifies to

$$r^2 + r - 2 = 0.$$

Notice that $L[e^{rt}] = (r^2 + r - 2)e^{rt}$, so the exponential is an 'eigenfunction' (analogous to an eigenvector) for this operator, which is why the e^{rt} can cancel.

The equation for r has solutions $r = -1$ and $r = 2$, so we conclude

$$e^{-t}, e^{2t} \quad \text{are solutions.}$$

But L is linear, so if L sends both solutions to zero, any linear combination will also be sent to zero by L. Thus,

$$y(t) = c_1 e^{-t} + c_2 e^{2t} \text{ is a solution for any scalars } c_1, c_2. \tag{3}$$

It's not obvious at this point that we have found the general solution (it is!).

One clue is that since a first order ODE needs one initial condition ($y(t_0) = y_0$) to have a unique solution, a second order ODE should need two, e.g.

$$y'' + y' - 2y = 0, \quad y(0) = a, \quad y'(0) = b.$$

Can this problem be solved given our solution? Plugging in, we get

$$a = c_1 + c_2, \qquad b = -c_1 + 2c_2$$

$$\implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which indeed has a solution for any (a, b) since the matrix is invertible. Thus, if our assertion about 'how many initial conditions are needed' is true, the solution (3) must be the general solution.

There are a number of hints to insights here - you can see structure here like a basis for a 2d space, linear independence, linearity and so on. That is thr structure we will identify in learning how to solve ODEs of this type.

3.2 Systems of ODEs, briefly

While our focus is on second-order ODEs, this context willl be useful.
A **first-order, linear**, system in n-dimensions has the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \tag{4}$$

where $\mathbf{x}(t)$ (the unknown) and $\mathbf{f}$ (given) are functions from $\mathbb{R}$ to $\mathbb{R}^n$ and $A(t)$ is an $n \times n$ matrix-valued function (i.e. $A(t)$ is a matrix for each t).

The 'initial value problem' specifies the vector $\mathbf{x}(t)$ at a time t_0:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{v}.$$

Here is an example of such an IVP in two dimensions:

$$\begin{cases} x_1' = 2x_1 + t^2 x_2 + \cos t \\ x_2' = \sin(t)\, x_1 + x_2 \end{cases} \qquad x_1(0) = 3, \quad x_2(0) = 7.$$

In matrix form (4), the system is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & t^2 \\ \sin t & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \cos t \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

The following version of the existence/uniqueness theorem is true:

Theorem (existence/uniqueness, linear systems) The initial value problem for

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}, \quad \mathbf{x}(t_0) = \mathbf{v}$$

has a unique solution, defined in the interval where $A(t)$ and $\mathbf{f}(t)$ are continuous.

Key point (2nd order ODEs are systems!): A second order ODE

$$y'' + p(t)y' + q(t)y = f(t) \tag{O}$$

is really a **first-order system in disguise**. Let

$$x_1 = y, \qquad x_2 = y'.$$

That is, we now have an unknown vector-valued function $(x_1(t), x_2(t)) = (y(t), y'(t))$. Then

$$\begin{aligned} x_1' &= x_2' \\ x_2' &= f(t) - p(t)x_2 - q(t)x_1 \end{aligned} \tag{S}$$

since $x_1' = y' = x_2$ and $x_2' = y'' = f(t) - p(t)y' - q(t)y$ from the ODE.

But (S) is just a first-order system for (x_1, x_2) equivalent to the second-order ODE (O).
That is, (with the $\iff$ meaning equivalent to)

$$\text{second order ODEs for } y(t) \iff \text{first order systems for } (y(t), y'(t))$$

Tis 'system for (y, y')' idea will be a useful notion shortly.

3.3 General theory for second-order linear ODEs

Now we consider the general[2] second-order linear **homogeneous** ODE

$$y'' + p(t)y' + q(t)y = f(t) \tag{5}$$

Defining the operator

$$Ly = y'' + py' + qy$$

the ODE can be written in the form

$$Ly = f.$$

From the previous section, identifying the ODE as a system for

$$x_1 = y, \quad x_2 = y'$$

we would have an IVP

$$\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_)) = \mathbf{x}_0$$

for an initial vector $\mathbf{x}_0$. Translating back to the second order ODE, it follows that the IVP should specify y and y' at t_0, so the right form of an IVP is

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = a, \quad y'(t_0) = b. \tag{6}$$

It follows that:

Theorem (existence/uniqueness, again) The second order, linear initial value problem

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = a, \quad y'(t_0) = b$$

has a unique solution, defined where the coefficients are continuous functions of t.

The important idea here is that **two conditions are required** to get a unique solution to a **second** order ODE.

As you may imagine, the pattern continues, with n conditions for an n-th order ODE.

[2]Note that one could also put a coefficient on y''; it is just omitted for simplicity.

4 Homogeneous problems, basis for solutions

Now let's look just at the homogeneous problem

$$y'' + p(t)y' + q(t)y = 0 \tag{H}$$

and let $Ly = y'' + p(t)y' + q(t)y$ (the operator).

Our goal is to understand the set of all solutions,

$$V = \{y : Ly = 0\}$$

and to verify that it is a vector space of dimension two.

4.1 Superposition (why is it a vector space?)

First, observe that if y_1 and y_2 are solutions, then so is $c_1y_1 + c_2y_2$ for any scalars since

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = c_1 \cdot 0 + c_2 \cdot 0.$$

It follows that the set of solutions V really is a vector space.

This observation is a powerful idea that we'll use often, so it gets a name:

Idea (superposition): If y_1 and y_2 are solutions to the **linear homogeneous** ODE (H) and c_1, c_2 are scalars, then the linear combination (or 'superposition')

$$c_1y_1 + c_2y_2$$

is also a solution. That is, solutions can be scaled and added together to form new solutions.

This is called the **principle of superposition**.

4.2 Preliminaries for the basis (linear independence)

Before finding a basis, we need a notion of what it means to be 'linearly independent' for functions.

For vectors, the useful linear combinations are spans of **linearly independent** vectors.

The question here is: what does it mean for **solutions to an ODE** to be linearly independent?

The motivating example from earlier (subsection 3.1) gives us the idea.

Consider the space of solutions to the homogeneous problem (H). Our goal is to be able to solve initial value problems of the form

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = a,\ y'(t_0) = b$$

for any initial conditions (a, b).

Suppose we have a solution $y_1(t)$. A solution $y_2(t)$ should be linearly independent if the linear combinations

$$c_1 y_1 + c_2 y_2$$

provide us with more solutions that solve IVPs (not just the ones covered by y_1).

From superposition, this is true if

$$\begin{bmatrix} y_1(t_0) \\ y_1'(t_0) \end{bmatrix}, \begin{bmatrix} y_2(t_0) \\ y_2'(t_0) \end{bmatrix} \text{ are linearly independent vectors for at least one } t_0. \qquad \text{(LI1)}$$

If true, we can combine the ICs for y_1 and y_2 to get any initial condition (a, b)..
Then by superposition, plus the existence theorem, we have constructed all the solutions.

On the other hand - what is the failure case? y_1 and y_2 will be linearly dependent if

$$\begin{bmatrix} y_1(t_0) \\ y_1'(t_0) \end{bmatrix}, \begin{bmatrix} y_2(t_0) \\ y_2'(t_0) \end{bmatrix} \text{ are linearly dependent for at one time} t_0.$$

That is, if at any time t_0, these IC vectors are linearly dependent, then $c_1 y_1 + c_2 y_2$ does not yield new solutions. This is the case since then

$$\begin{bmatrix} y_2(t_0) \\ y_2'(t_0) \end{bmatrix} \text{ is in span}(\begin{bmatrix} y_1(t_0) \\ y_1'(t_0) \end{bmatrix})$$

so we really only have solutions that start at multiples of $(y_1(t_0), y_1'(t_0))$ (so just cy_1).

This means that y_1 and y_2 should be linearly independent only if

$$\begin{bmatrix} y_1(t_0) \\ y_1'(t_0) \end{bmatrix}, \begin{bmatrix} y_2(t_0) \\ y_2'(t_0) \end{bmatrix} \text{ are linearly independent at } \textbf{all times } t_0. \qquad \text{(LI2)}$$

to avoid this failure case.

The argument can be made rigorous to show these two definitions are actually equivalent.

Definition: Two solutions to the linear, homogeneous ODE (H) are said to be **linearly independent** if (LI1) holds, or equivalently if the stronger statement (LI2) holds.

This means that for a pair of solutions, if the vectors (y, y') are linearly independent at one time, they are linearly independent at all times!

Obviously, the at one time condition is the easier one to check in practice.

That is, we need only show linear independence of the vectors (y_1, y_1') and (y_2, y_2') at a single point; then the lemma tells us these vectors are linearly independent at all t. Recalling some linear algebra, (LI1) holds if and only if

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \neq 0.$$

This expression is called the **Wronskian**, defined to be

$$W(y_1, y_2)(t) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2.$$

Thus y_1, y_2 is a basis if and only if W is non-zero at some point:

$$\{y_1, y_2\} \text{ is a basis } \iff W(y_1, y_2)(t_0) \neq 0 \text{ for at least one } t_0.$$

It follows from the lemma that

$$W(t_0) \neq 0 \text{ at some } t_0 \iff W(t) \neq 0 \text{ for all t.}$$

The test is useful, but you should keep in mind that it comes from (LI1).

Example (checking linear independence, plus a bonus solution guess):

$$L[y] := t^2 y'' - 2y = 0, \quad t > 0.$$

By some inspired guesswork, let's try a solution of the form $y = t^r$. Substituting in, we find that

$$L[t^r] = (r^2 - r - 2)t^r.$$

Thus t^r is a solution if and only if $r = 2$ or $r = -1$. We therefore have two solutions

$$y_1 = t^2, \quad y_2 = 1/t.$$

Obviously, y_1 is not a multiple of y_2, which establishes that they are a basis by (**??**). To verify directly, we check linear independence of (y_1, y_1') and (y_2, y_2'):

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} t^2 & 1/t \\ 2t & -1/t^2 \end{bmatrix}.$$

By the previous results, it suffices to check at a single point. Take $t = 1$; then

$$(1, 2), \quad (1, -1) \text{ are linearly independent}$$

so it follows that y_1, y_2 are a basis for solutions to $Ly = 0$.

The determinant (which is the Wronskian) at t is

$$W(t) = -3.$$

Note that $W(1) = -3$, which is what we used above. Indeed, it is true that $W(t) \neq 0$ everywhere, as it must be if it is non-zero at a point. It is, of course, much easier to show that $W(t_0) \neq 0$ for one t_0 than to show that $W \neq 0$ for all t.

4.3 Constructing the basis

We have now shown that the set of solutions to

$$y'' + p(t)y' + q(t)y = 0 \tag{7}$$

is a **vector space** of dimension two, i.e.

$$\{\text{solutions to } (7)\} = \text{span}(y_1, y_2)$$

for a pair of basis solutions y_1, y_2. To show this, start with the fact that

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = a,\ y'(t_0) = b$$

has a solution for any (a, b). We can choose any two LI sets of initial conditions and get two LI solutions. For instance, let

$$y_1 = \text{ solution to the ODE with } y(t_0) = 1,\ y'(t_0) = 0,$$

$$y_2 = \text{ solution to the ODE with } y(t_0) = 0,\ y'(t_0) = 1$$

Then

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad W(t_0) = \det(\cdots) = 1$$

which confirms the solutions are linearly independent and form a basis for solutions (this is also called a **fundamental set**).

It's clear, then, that any solution to the ODE is a linear combination of these solutions, since we need only combine them to match the values at t_0

$$y(t) \text{ solves the ODE } \implies y(t) = y(t_0)y_1(t) + y'(t_0)y_2(t).$$

Unfortunately, we don't have a general method for finding such solutions - but if we *can* do so, that solves the ODE and lets us solve any initial value problem.

5 Constant-coefficient, second-order linear ODEs

If the coefficients are constant, then we do have a way to solve the ODE exactly. The procedure is fairly straightforward, but there are a few details that require care.

These ODEs are extremely common, so it is essential to know how to solve them and to be able to efficiently compute the solutions!

We cosnider the linear, second order, homogeneous, **constant coefficient** ODE (take a moment to recall what each words means in this long list!)

$$ay'' + by' + cy = 0. \tag{C}$$

Assume also that the coefficients a, b, c are **real numbers**.
This type of ODE will be abbreviated as 'LCC' **(Linear Constant Coefficient).**

As before, let $Ly = ay'' + by' + cy$ denote the associated operator.
By the previous discussion, it suffices to:

- Find two solutions
- Check that they are linearly independent
- Take linear combinations to get the general solution

5.1 Procedure

The procedure to find the two linearly independent solutions is listed below (examples listed separately afterwards). Note that with these two solutions, we are done, as they span the set of all solution to the ODE (C).

1) Find exponential solutions: Look for solutions of the form

$$y(t) = e^{\lambda t}.$$

Observe that (check this!)

$$L[e^{\lambda t}] = (a\lambda^2 + b\lambda + c)e^{\lambda t}$$

so it follows that

$$e^{\lambda t} \text{ is a solution} \iff 0 = a\lambda^2 + b\lambda + c$$

The function $p(\lambda) = a\lambda^2 + b\lambda + c$ is called the **characteristic polynomial**.

Now let

$$\lambda_1,\ \lambda_2 = \text{ roots of the characteristic polynomial.} \tag{8}$$

There are two solutions $e^{\lambda_1 t}, e^{\lambda_2 t}$ if they are distinct, and only one solution $e^{\lambda t}$ if they are equal. Then, to get the actual basis solutions...

2) Construct the basis solutions: There are three cases to consider here.

i) **Distinct real roots:** If $\lambda_1 \neq \lambda_2$ and both are **real**, then we have a pair of solutions

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = e^{\lambda_2 t}.$$

You can check that they are linearly independent.

ii) **Complex roots:** Since a, b, c are real, the roots of the characteristic polynomial (8) must be complex conjugates in the form

$$\lambda = r \pm i\omega.$$

Step (1) does give us two exponential solutions, but they are complex:

$$e^{(r+i\omega)t}, \quad e^{(r-i\omega)t}.$$

These are a perfectly fine basis for solutions, but would require allowing *complex coefficients*, and ending up with all *complex solutions*:

$$y(t) = e^{rt}\left(z_1 e^{i\omega t} + z_2 e^{-i\omega t}\right), \qquad z_1, z_2 \in \mathbb{C}.$$

To get a **real** basis for **real** solutions, we use the fact that

$$y(t) \text{ is a solution } \implies \text{the real/imaginary parts are solutions.}$$

(see homework for details - this relies on linearity and the real coefficients a, b, c).

It follows that each complex solution yields two real solutions. Taking the real/imaginary parts of either one gives

$$y_1 = \text{Re}(e^{(r+\omega i)t}) = e^{rt}\cos\omega t, \quad y_2 = \text{Im}(e^{(r+\omega it)}) = e^{rt}\sin\omega t$$

Again, you can check that these solutions are linearly independent.

iii) **Repeated roots:** This case is more trouble. We have only one exponential solution

$$y_1 = e^{\lambda t}.$$

The rule here is to **multiply by t to get a new solution**. A second solution is

$$y_2 = te^{\lambda t}.$$

You can check that (i) y_2 really is a solution and (ii) y_1, y_2 are linearly independent. A quick proof of the t rule is given below. The next set of notes also gives a way to show the result (reduction of order).

5.2 Some comments on the LCC procedure

Existence: The existence theorem guarantees that, since the coefficients a, b, c are all continuous in t, solutions must exist for all t.

This can be seen also by looking at the solutions. The functions that show up are:

$$e^{\lambda t}, \qquad e^{\lambda t}(\sin t \text{ or } \cos t), \quad te^{\lambda t}$$

are all well-defined for all t. Moreover, the type of behavior this ODE can display is quite limited - it can only be linear combinations of a pair of the above.

Repeated roots case: Suppose r is the repeated root for

$$y'' + by' + cy = 0.$$

We know that, for **any value** λ,

$$L[e^{\lambda t}] = p(\lambda)e^{\lambda t} = (\lambda - r)^2 e^{\lambda t}$$

by factoring the characteristic polynomial. Differentiate in λ (not in t!) to get

$$\frac{\partial}{\partial \lambda}\left(L[e^{\lambda t}]\right) = \frac{\partial}{\partial \lambda}\left((\lambda - r)^2 e^{\lambda t}\right)$$
$$\implies L[te^{\lambda t}] = 2(\lambda - r)e^{\lambda t} + t(\lambda - r)^2 e^{\lambda t}$$

since the λ partial derivative can be moved inside the L (why?).

Now plug in $\lambda = b$, and we find that

$$L[te^{bt}] = 0 \implies te^{bt} \text{ is a solution.}$$

The repeated root preserves the evaluates-to-zero-at-b property of the RHS after taking a derivative in λ, which lets us get a a new solution by taking

$$y_2 = \frac{\partial}{\partial \lambda}(e^{\lambda t}).$$

Note that this trick only generalizes to higher order LCC ODEs (see subsection 5.4), but otherwise isn't useful - it's specific to this repeated roots, LCC case.

5.3 A series of examples

Examples accompanying the procedure, plus a few typical calculations and observations.

Example (i) - real roots: Solve

$$y'' - 9y = 0, \qquad y(2) = 1, \quad y'(2) = 0.$$

THe characteristic polynomial/roots are

$$p(\lambda) = \lambda^2 - 9, \quad \lambda = \pm 3.$$

The general solution to the ODE is then

$$y(t) = c_1 e^{3t} + c_2 e^{-3t}.$$

Plugging this into the initial caonditions we get

$$1 = c_1 e^6 + c_2 e^{-6}, \qquad 0 = 3c_1 e^6 - 3c_2 e^{-6}$$

which gives us $c_1 e^6 = c_2 e^{-6} = 1/2$ and so

$$\begin{aligned} y(t) &= \frac{1}{2} e^{-6} e^{3t} + \frac{1}{2} e^{6} e^{-3t} \\ &= \frac{1}{2} e^{3(t-2)} + \frac{1}{2} e^{-3(t-2)}. \end{aligned}$$

Remark (translation is convenient!): The example hints at the fact that is is often good to **center the basis at the initial** t. That is, write $e^{\lambda(t-t_0)}$ or $\sin(\omega(t-t_0))$ so that they evaluate to nicer quantities (no e's!) at $t = t_0$.

We can do this because the LCC ODE is **autonomous**, so

$$y(t) \text{ is a solution } \implies y(t - t_0) \text{ is also a solution}$$

i.e. translations in t of $y(t)$ are also solutions.

In the example above, an ideal choice of basis is $y_1 = e^{3(t-2)}$ and $y_2 = e^{-3(t-2)}$, since then $y = c_1 y_1 + c_2 y_2$ with

$$y(2) = 1 \implies c_1 + c_2 = 1, \quad y'(2) = 0 \implies 3c_1 - 3c_2 = 0.$$

Example (ii) - complex roots: The equation

$$y'' + \omega^2 y = 0, \qquad \omega \in \mathbb{R}$$

dscribes **simple harmonic oscillators** (e.g. a vibrating spring with no friction).

The characteristc polynomial and roots are

$$p(\lambda) = \lambda^2 + \omega^2, \qquad \lambda = \pm\omega i,$$

The complex basis solutions are then $e^{\pm i\omega t}$. Taking real and imaginary parts yields solutions $\sin(\omega t)$ and $\cos(\omega t)$ so the real solution is

$$y(t) = c_1 \sin \omega t + c_2 \cos \omega t.$$

The solution oscillates with frequency ω (with a fixed amplitude). This system is the simplest autonomous ODE that has oscillating solutions.

Example (iii) - repeated roots): Consider the IVP

$$4y'' - 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 3/2.$$

The characteristic polynomial is $p(\lambda) = (2\lambda - 1)^2$ so two linearly independent solutions are $e^{t/2}$ and $te^{t/2}$. The general solution is

$$y = (c_1 t + c_2)e^{t/2}.$$

Plugging in $y(0) = 1$ we get $c_2 = 1$; then $y'(0) = 3/2$ gives $c_1 = 1$.

Note that it's easier here to solve for c_2, c_1 in succession.

Example (iv) - choice of basis: We solve the IVPs

$$\text{(A)}: \quad y'' - 9y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

$$\text{(B)}: \quad y'' - 9y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

From example (i), the general solution to the ODE can be written as

$$y(t) = c_1 e^{3t} + c_2 e^{-3t}.$$

Now for the IVP (A) , plugging in the ICs gives

$$\begin{cases} 1 = c_1 + c_2, \\ 0 = 3c_1 - 3c_2 \end{cases} \implies \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which is easily solved to get $c_1 = c_2 = 1/2$.
For (B), the equations are almost the same:

$$\begin{cases} 0 = c_1 + c_2, \\ 1 = 3c_1 - 3c_2 \end{cases} \implies \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $c_1 = 1/2$ and $c_2 = -1/2$. Thus we have two solutions

$$y_a = \frac{1}{2}(e^{3t} + e^{-3t}) = \cosh(3t), \qquad y_b = \frac{1}{2}(e^{3t} - e^{-3t}) = \sinh(3t).$$

Now note that the initial conditions are $(1, 0)$ and $(0, 1)$. Thus

$$(y_a, y_a') \text{ and } (y_b, y_b') \text{ are LI vectors at } t = 0$$

(or equivalently, $W(0) = 1$ if you want to use the Wronskian), so

$$y_a, y_b \text{ are LI solutions.}$$

This means that we can write the general solution as

$$y(t) = c_1 \cosh(3t) + c_2 \sinh(3t).$$

Notably, because of the initial conditions being nice,

$$y(t) = y(0) \cosh(3t) + y'(0) \sinh(3t).$$

That is, this choice makes solving for the coeffiicents using ICs at $t = 0$ trivial: the first term vanishes for $y'(0)$ and the second term vanishes for $y(0)$.

From time to time, we will use this 'sinh/cosh' basis instead for convenience.

5.4 Higher order ODEs

The procedure is the same for higher-order LCC ODEs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

with corresponding initial value problems specifying values

$$y(t_0) = b_0, \quad y'(t_0) = b_1, \ \cdots \ y^{(n-1)}(t_0) = b_{n-1}.$$

(these ODEs are really first order systems for $(y, y', \cdots, y^{(n-1)})$).

The space of solutions is now spanned by n basis functions, so we need to obtain n solutions from the procedure.

First, plug in e^{rt} to find that

$$e^{rt} \text{ is a solution } \iff p(r) = 0$$

where $p(r)$ is a characteristic polynomial of degree n (check this!).

Now the procedure is extended slightly:

- Each non-repeated real root gives a solution e^{rt}
- Each pair of complex roots gives two solutions as in the 2nd order case
- Each root r repeated k times gives solutions

$$e^{rt}, te^{rt}, \cdots, t^{k-1}e^{rt}.$$

That is, we multiply by t to get new solutions, once for each repetition.

The last step takes some effort to show, but the rest is just as in the second order case.

Examples: For the equation

$$y''' - y'' - y' + 1 = 0,$$

we have

$$p(r) = (r-1)^2(r+1), \quad r = -1, 1, 1$$

so e^{-t}, e^t and te^t are the basis solutions provided by the procedure.

The general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^t + c_3 te^t.$$

The equation

$$y^{(4)} - y = 0, \qquad y(0) = 1, \quad y'(0) = y''(0) = y'''(0)$$

has characteristic polynomial/roots

$$p(r) = r^4 - 1, \quad r = \pm 1, \pm i.$$

The two complex roots give two real solutions $\sin t$ and $\cos t$, and the general solution is

$$y = c_1 \sin t + c_2 \cos t + c_3 e^t + c_4 e^{-t}.$$

Plugging in the initial conditions,

$$1 = c_2 + c_3 + c_4, \quad 0 = c_1 + c_3 - c_4, \quad 0 = -c_2 + c_3 + c_4, \quad 0 = -c_1 + c_3 - c_4$$

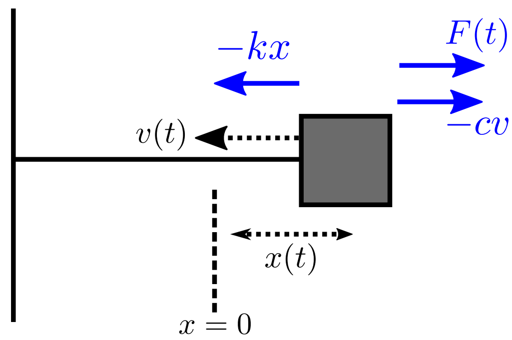
which can be solved (first get $c_2 = 1/2$, then $c_3 = c_4$) to obtain

$$c_1 = 0, \quad c_2 = \frac{1}{2}, \quad c_3 = c_4 = \frac{1}{4}$$

so the solution is

$$y(t) = \frac{1}{2}\cos t + \frac{1}{4}e^t + \frac{1}{4}e^{-t}.$$

6 Example: a spring



Consider a spring with displacement $x(t)$ from $x = 0$ and velocity $v(t)$.
Suppose the spring is influenced by the following forces:

- A spring force $-kx$
- A damping force $-cv$
- An external force $F(t)$ applied to the spring

We'll deal with the $F(t)$ part later. For now, assume that the spring is stretched to some initial displacement and then let go, allowed to vibrate (?).

According to Newton's third law,

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}.$$

Upon rearranging (and letting x' denote dx/dt etc.), we get

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0.$$

Let's see what the solution says about the spring's behavior. FIrst of all, note that

$$\text{no damping} \implies x'' + \frac{k}{m}x = 0$$

which has solutions

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t, \qquad \omega = \sqrt{k/m}$$

i.e. the spring oscillates at a frequency ω.

What happens with damping? For simplicity, let's take the equation

$$x'' + 2\beta x' + x = 0$$

instead (this turns out to be sufficient to study - see the homework).

The characteristic polynomial has roots

$$\lambda_1, \lambda_2 = -\beta \pm \sqrt{\beta^2 - 4}.$$

First, notice that if $\beta > 2$ then both roots are real and negative; the solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Both terms go to zero as $t \to \infty$ regardless of c_1, c_2 so

$$\lim_{t\to\infty} x(t) = 0.$$

Moreoever, there is **no oscillation**. The spring just shrinks back to its resting state (we call this **overdamped**); the damping is strong enough that it prevents any back-and-forth-motion.

The rest of the cases are left to you. It should be true that if there is **positive** damping ($\beta > 0$), then

$$\lim_{t\to\infty} x(t) = 0 \text{ for any initial condition}$$

since the spring should lose energy from the damping over time (the force acts in the opposite direction of its motion).