Math 353 Lecture Notes
First order ODEs: exact equations

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Topics covered

- Exact equations and differentials
- Connection to conservative vector fields

1 Exact equations

1.1 Background/context

For the purely mechanical treatment of exact equations, see the textbook. Instead, we will motivate the study of exact equations by making a detour into vector calculus, and arrive naturally at the procedure for solving exact equations.

(Review: conservative vector fields) Recall that a vector field

$$\mathbf{v} = (M(x,y), N(x,y))$$

defined in a nice[1] region D is *conservative* if it is the gradient of a 'potential $\phi(x,y)$:

$$\mathbf{v} = \nabla\phi.$$

The gradient theorem states that a vector field is conservative if and only if it has path independence, i.e. the line integral of $\mathbf{v}$ along any path γ from a point a to b is the same:

$$\phi(b) - \phi(a) = \int_\gamma M\,dx + N\,dy$$

Further, this holds if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{in } D. \tag{P}$$

Note that the line integral above, for an explicit path $(x(t), y(t))$ from $t = a$ to $t = b$, is

$$\int_\gamma M\,dx + N\,dy = \int_a^b \left(M(x(t), y(t))\frac{dx}{dt} + N(x(t), y(t))\frac{dy}{dt} \right) dt$$

A key question is how to 'go in reverse': Suppose we have a vector field

$$\mathbf{v} = (M(x, y), N(x, y)).$$

How do we compute the potential ϕ?

Step 0: Befsore starting, check that the vector field really is conservative by using the condition (P).

Next, observe that we arre looking for a function ϕ that satisfies

$$M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}.$$

Both equations can be integrated to get information about ϕ.

Step 1: Integrate the M-equation with respect to x and obtain

$$\phi = (\text{func. of } x \text{ and } y) + h(y).$$

Note that an arbitrary $h(y)$ must be included, not just a constant, since that is the most general function of (x, y) that is sent to zero by $\partial/\partial x$.

Step 2: Now solve for $h(y)$. To do this, plug the expression for ϕ into the N-equation:

$$\frac{\partial \phi}{\partial y} = N \implies \frac{\partial}{\partial y}(\cdots) + h'(y) = N.$$

This equation has one unknown, $h(y)$, that can now be found.

Note that Step 2 only works if the above reduces to

$$\text{function of } y \text{ only} = h'(y).$$

However, it can be shown that as long as (P) holds (i.e. the vector field is conservative), all the x's will cancel as they must!

Example: We find the potential ϕ for the vector field

$$\mathbf{v} = (2xy, 2y + x^2).$$

First, check that this is conservative using (P)

$$\frac{\partial}{\partial y}(2xy) = 2x, \quad \frac{\partial}{\partial x}(2y + x^2) = 2x.$$

Now

$$\frac{\partial \phi}{\partial x} = 2xy \implies \phi = \int 2xy\,dx = x^2 y + h(y).$$

Plug this into the other equation:

$$\frac{\partial \phi}{\partial y} = 2y + x^2 \implies x^2 + h'(y) = 2y + x^2.$$

Conveniently, the x^2's cancel, leaving an equation to solve for h:

$$h'(y) = 2y \implies h(y) = y^2 + C$$

Thus, the potential is

$$\phi = x^2 y + y^2 + C.$$

Note that the potential is unique only up to the constant C, which it must be since only derivatives of ϕ appear in the defining equations.

Example: Consider the vector field

$$\mathbf{v} = (2x\cos y,\ x^2 \sin y).$$

Checking the condition,

$$\frac{\partial M}{\partial y} = -2x\sin y, \quad \frac{\partial N}{\partial x} = 2x\sin y$$

so this field is not conservative!

Now if we were to try the method anyway... From the M-equation,

$$\phi = \int (2x\cos y)\,dx = x^2\cos y + h(y).$$

Then

$$\frac{\partial \phi}{\partial y} = x^2\sin y \implies -x^2\sin y + h'(y) = x^2\sin y$$

which would require solving

$$h'(y) = 2x^2\sin y.$$

But $h(y)$ is only a function of y, so this equation cannot be solved.

1.2 Exact equations

The solution procedure can be used to solve certain ODEs as well.

Suppose we have a potential ϕ and, as before,

$$\nabla\phi = (M(x,y), N(x,y)).$$

A curve $(x, y(x))$ of 'constant potential' is one that satisfies

$$\phi(x, y(x)) = C \tag{C}$$

such as the arcs of a circle for $\phi = x^2 + y^2$ we saw earlier.

Taking the derivative gives an ODE for $y(x)$, Using the chain rule:

$$\frac{d}{dx}(\phi(x, y(x))) = 0 \implies \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0.$$

Thus, curves of constant potential are solutions to the ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \tag{E}$$

When (M, N) come from a potential ϕ, this is called an **exact equation**.

Solution procedure: Solving an exact ODE is just a matter of finding the potential. The solution to the ODE (E), when exact, is just the curve of constant potential (C). This is the implicitly defined solution, and it can then be solved for $y(x)$ as needed.

To be explicit:

- Write the equation in the 'chain rule'-d form

  $$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

  anticipating that it may really be $\frac{d}{dx}(\phi(x,y)) = 0$.

- Check that (M, N) is conservative using the condition. If it is not, then you are out of luck (try another method).

- If it is conservative, find the potential ϕ such that

  $$\frac{\partial\phi}{\partial x} = M, \quad \frac{\partial\phi}{\partial y} = N.$$

- The solution then satisfies $\phi(x,y) = C$.

Example: Suppose we need to solve the ODE

$$2x - y + (2y - x)y' = 0.$$

Check that it is exact:

$$\frac{\partial}{\partial y}(2x - y) = -1, \quad \frac{\partial}{\partial x}(2y - x) = -1.$$

Now we find ϕ such that

$$\frac{\partial \phi}{\partial x} = 2x - y, \quad \frac{\partial \phi}{\partial y} = 2y - x.$$

You can integrate either one first. For variety, start with the y-equation:

$$\phi = y^2 - xy + h(x).$$

Plug this into the x equation to get

$$\frac{\partial \phi}{\partial x} = 2x - y \implies -y + h'(x) = 2x - y.$$

This gives $2x = h'(x)$, so $h(x) = x^2$ and

$$\phi = x^2 - xy + y^2.$$

Solutions $y(x)$ to the ODE then satisfy

$$x^2 - xy + y^2 = C.$$

Solutions to the ODE are segments/arcs of an ellipse (see figure below).

Continued (an initial value problem): Now we solve the IVP

$$2x - y + (2y - x)y' = 0, \qquad y(0) = \sqrt{3}/2.$$

Solving for the constant yields the implicit solution

$$x^2 - xy + y^2 = \frac{3}{4}.$$

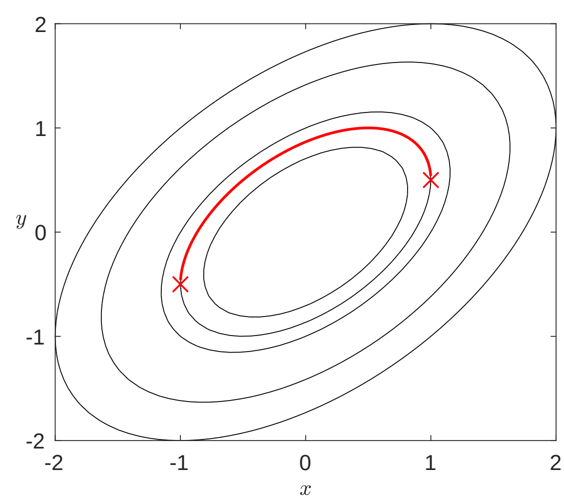
The interval of existence can be found by looking for points where $|y'| \to \infty$. From the ODE, this occurs when $y = x/2$ since

$$y' = \frac{y - 2x}{2y - x}.$$

Plugging $y = x/2$ into the solution yields

$$3x^2/4 = 3/4 \implies x = \pm 1.$$

It follows that the solution is defined for $(-1, 1)$.



1.3 A few more notes (exact equations)

Geometry: Observe that if

$$\phi(x, y(x)) = C$$

and ϕ is conservative with $\nabla\phi = (M, N)$, then

$$M + N\frac{dy}{dx} = 0 \implies (M, N) \cdot (1, y'(x)) = 0.$$

But $(1, y'(x))$ is the direction of the tangent.
Thus, $y(x)$ is **perpendicular** to the vector field. This means that exact equations describe curves that run perpendicular to the vector field (M, N).

For example:

$$\phi = \frac{1}{2}(x^2 + y^2), \qquad \nabla\phi = (x, y)$$

and the curves perpendicular to (x, y) go in circles. This sort of motion shows up often in physics, where motion can be driven by some potential (energy).

Exact differentials: An exact equation is often written as

$$M\,dx + N\,dy = 0.$$

This is called an **exact differential**, and is essentially shorthand, similar to the notation

$$f(y)\,dy = g(x)\,dx$$

for separable equations (in fact, this is a special case!). To be precise, the notation

$$d\phi = M\,dx + N\,dy$$

means that (M, N) is conservative with potential ϕ. It indicates that along a curve $(x(t), y(t))$, one can 'divide by dt' to get

$$\frac{d}{dt}(\phi(x(t), y(t)) = M\frac{dx}{dt} + N\frac{dy}{dt}$$

or alternatively given $y(x)$ one can 'divide by dx' to get

$$\frac{d}{dx}(\phi) = M + N\frac{dy}{dx}$$

which is the expression we encountered before.