Math 353 Lecture Notes
First order ODEs
(Theory and qualitative behavior)

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Topics covered

- Existence and uniqueness of solutions to first-order ODEs
  - Conditions for local existence/uniqueness
  - How uniqueness constrains solution behavior
  - A physical example of non-uniqueness
- Flow on a line (autonomous first-order ODEs)
  - Equilibria and stability
  - Analysis using phase lines
  - Modeling example

1 Existence and uniqueness

Consider a general first-order initial value problem

$$y' = f(t,y), \qquad y(t_0) = y_0.$$

There are three fundamental questions to address:

- When do solutions to this problem exist, at least near the initial value?
- When is this solution unique?
- How far do solutions extend when they do exist?

The following examples illustrate why the answers might be non-trivial:

$$y' = y, \quad y(0) = y_0 \tag{a}$$
$$y' = ty^2, \quad y(0) = y_0 > 0 \tag{b}$$
$$y' = y/t, \quad y(0) = y_0 \neq 0 \tag{c}$$
$$y' = 2y^{1/2}, \quad y(0) = 0. \tag{d}$$

Let's solve the equations exactly and answer the questions for each.

a) Using an integrating factor, the ODE is equivalent to

$$(e^{-t}y)' = 0.$$

Integrating, we get the general solution

$$y = Ce^t,$$

and so $y(t) = y_0 e^t$. Notice that the steps in the derivation are all 'if and only if' (why?), so every solution to the ODE *must* have the form Ce^t. Thus the solution to the IVP exists, is unique, and is defined on all of $\mathbb{R}$.

b) The ODE is separable; the general solution is

$$y(t) = \frac{1}{C - t^2}$$

and the IVP has a unique solution $y(t) = (1/y_0 - t^2)^{-1}$ which blows up at $t = \pm 1/\sqrt{y_0}$. Thus the solution is only defined in the interval $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$, which can be arbitrarily small depending on y_0.

c) Separating variables, we find that $y = Ct$ is a solution (for any C). If $y_0 \neq 0$ there is no solution (since $y(0) = 0$ for any C). If $y_0 = 0$ then any C works, so $y = Ct$ is a solution for any C; there are infinitely many solutions.

d) This is the most subtle example - the ordinary methods of solution are not sufficient. The general solution by separating variables is

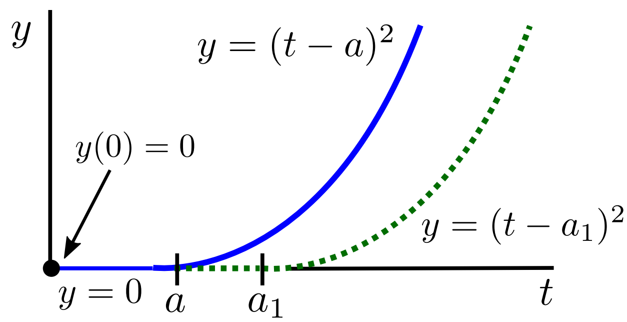
$$y(t) = (t - a)^2 \tag{1}$$

where a is a constant (that can take any value); t_c is the time at which $y(t)$ has its minimum. Applying the initial condition, we find that $y(t) = t^2$ is a solution.

However, observe that $y(t) = 0$ is also a solution - which means there are at least two. But there are more! Observe that the parabolic solution (1) has $y'(a) = 0$, which means we can always glue the increasing part of the parabola to $y = 0$ to form a piecewise solution:

$$y(t) = \begin{cases} 0 & t < a \\ (t-a)^2 & t \geq a. \end{cases}$$

This solves the IVP for **any** $a > 0$ (some examples are sketched below). That is, a solution to the IVP can start out as $y(t) = 0$ up to a time $a > 0$ of our choosing, at which point it suddenly becomes non-zero.



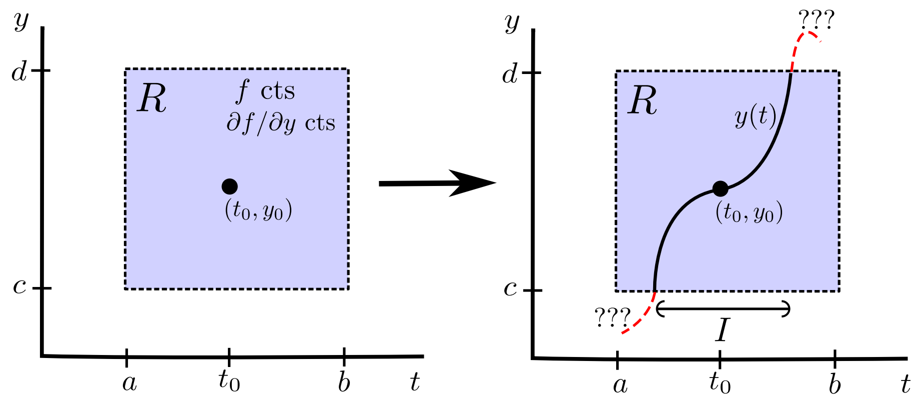
1.1 The existence theorem

The theorem below[1] provides some easy to check conditions that guarantee a unique solution exists near the initial point (t_0, y_0).

We consider the first-order IVP

$$y' = f(t, y), \qquad y(t_0) = y_0$$

and define a rectangle $R = (a, b) \times (c, d)$ in the (t, y) plane containing (t_0, y_0) (see diagram).



Theorem (existence) (i) If

$$f(t, y) \text{ is continuous in } R$$

then at least one local solution exists to the IVP, in a subinterval $I \subset (a, b)$ containing t_0.

(ii) If in addition,

$$\frac{\partial f}{\partial y} \text{ is continuous in } R$$

then this solution is unique.

[1]Part (i) of this theorem is the *Peano existence theorem*, and part (ii) is a simplified version of the important *Picard-Lindelof theorem*. The theorem as stated here can be improved to be a bit more precise, but is much more technical.

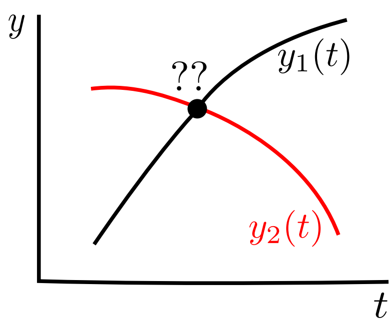
Continuity of f guarantees existence, and continuity of $\frac{\partial f}{\partial y}$ guarantees uniqueness. Note that we do *not* obtain a solution in all of the domain where the assumptions hold, only in a (possibly very small) sub-interval.

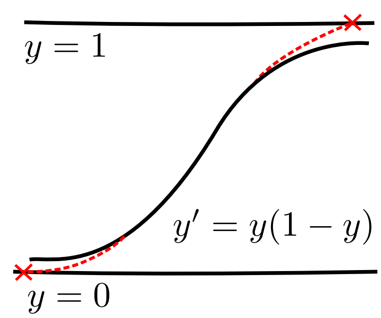
Applying this to our examples: (a) and (b) satisfy both conditions in all of the (t, y) plane, so a local solution exists and is unique. However in (b) the solution does not extend to the whole domain. The theorem cannot distinguish between the two cases, as it only guarantees a solution near the initial point.

In (c), condition (i) is violated; while (ii) is true trivially, it does not matter because we don't care if a solution is unique when it does not exist. Lastly, (d) satisfies (i) but not (ii), and indeed solutions are not unique.

1.2 Consequences: non-intersecting solutions

Uniqueness (part (ii) of the theorem) constrains where solutions can go if the conditions of the theorem are satisfied. The most important consequence is that *distinct* solution curves $(t, y_1(t))$ and $(t, y_1(t))$ in the (t, y) plane cannot intersect. If they did, then there would be two solutions to the IVP starting at the intersection point - a contradiction (see left figure).





1 We can sometimes use this to show bounds for solutions. As an example, observe that

$$y' = ty(1-y)$$

has constant solutions $y \equiv 0$ and $y \equiv 1$. Thus (see right figure above) any solution curve starting at a y-value between 0 and 1 must stay within those bounds for all time (at least for all times where it exists). That is, we have

$$0 < y_0 < 1 \implies 0 < y(t) < 1 \text{ for all } t.$$

No solution can cross the lines $y = 0$ and $y = 1$.

1.3 Non-uniqueness for a physical model:

The following example[2] shows how non-uniqueness can arise in a physical scenario.

A bucket filled with water has a hole in the bottom. The bucket has cross-sectional area A, the hole has area a, and the water has density ρ. At a time t, the height of water in the bucket is $h(t)$ and the velocity of the escaping water is $v(t)$. We want an ODE for $h(t)$.

In a small time Δt, a 'cylinder' of water with length $\rho v \Delta t$ (and area a) leaves, while $\rho A \Delta h$ water is lost in the bucket. Thus by conservation of mass,

$$av\Delta t = A\Delta h. \tag{2}$$

Now the system loses potential energy by an amount equivalent to moving the volume of lost water from the top of the bucket to the hole, i.e.

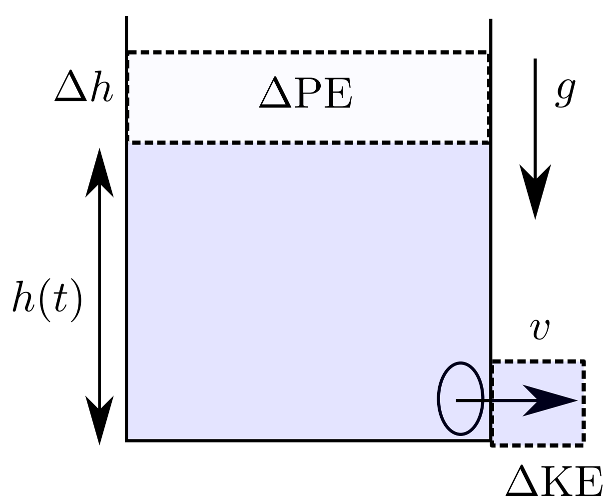
$$\Delta \text{P.E.} = \rho(A\Delta h)gh.$$

The kinetic energy given to the escaped water in the time Δt is

$$\Delta \text{K.E.} = \frac{1}{2}(\rho a v \Delta t)v^2.$$

Equating the two, we find obtain *Torricelli's law*:

$$v^2 = 2gh. \tag{3}$$



Combining Torricelli's Law (3) with (2), we obtain

$$\frac{dh}{dt} = -Ch^{1/2}, \qquad C = a\sqrt{2g}/A.$$

[2]From Strogatz' *Nonlinear Dynamics and Chaos*, Ch. 2.

If $h(0) = h_0 > 0$ then the unique solution is (by separating variables)

$$h(t) = \frac{C^2}{4} \left(\frac{4h_0}{C^2} - t \right)^2 .$$

This goes to zero in a finite time. But what happens if $h(0) = 0$, i.e. we start with an empty bucket? Clearly it is not possible to know when the bucket was full (it could have finished emptying at any time before 0). This is reflected in the fact that solutions are non-unique *backwards in time*. For any $t_c < 0$ the ODE has a solution

$$h(t) = \frac{C^2}{4} \begin{cases} (t_c - t)^2 & t < t_c \\ 0 & t > t_c. \end{cases}$$

corresponding to the scenario where the bucket was full at some point before t_c and then became completely empty at t_c, then stayed empty up to the present.

2 Geometric interpretation

Solutions can be visualized as curves in the (t, y) plane using a **direction field**, which sometimes gives useful intuition. The equation

$$y'(t) = f(t, y)$$

tells us the rate of change of y for a solution curve $(t, y(t))$ at any point (t, y). The solution curve is tangent to the vector field

$$\mathbf{v}(t, y) = (1, f(t, y)).$$

We draw this vector field (the 'direction field') as arrows (ignoring the magnitude). All solutions must 'follow' this direction field. The advantage is that it is straightforward to draw; the downside is that it provides limited information.[3]

Example 1:

$$y'(t) = y(1 - y).$$

The direction field is shown with solutions curves from three starting points at $t = 0$ (for $y(0) = -0.5, 0.5$ and 1.5). From the plot, we can deduce what solutions must do as t increases to ∞ or decreases to $-\infty$. By following the direction field, the plot suggests

$$y_0 > 0 \implies \lim_{t\to\infty} y(t) = 1$$
$$y_0 < 1 \implies \lim_{t\to-\infty} y(t) = 0.$$

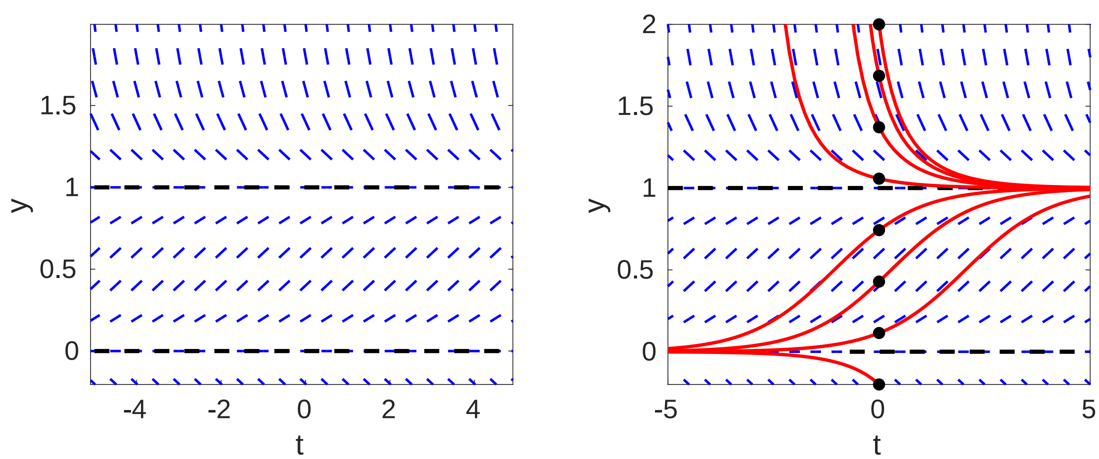
For the other cases, it appears that $y(t)$ diverges, but the behavior is not obvious from the direction field. As an exercise, you may determine from the exact solution the behavior for

$$y_0 > 1, \quad t \to -\infty \text{ or } y_0 < 0, \quad t \to \infty.$$

Equilibrium points: There are also two solutions that are constant:

$$y(t) \equiv 0 \text{ or } 1$$

which are **equilibrium points** (since the solution doesn't move). On the direction field, these are horizontal lines (where $y' = 0$ always. (We'll have more to say later).



[3]Drawing the field is tedious by hand; we'll address some more efficient and powerful tools in Part II of the course. It is easy to plot a direction field in Matlab - an example can be found on the course website

Example 2: Consider

$$y' = -ty + 1, \quad y(0) = y_0.$$

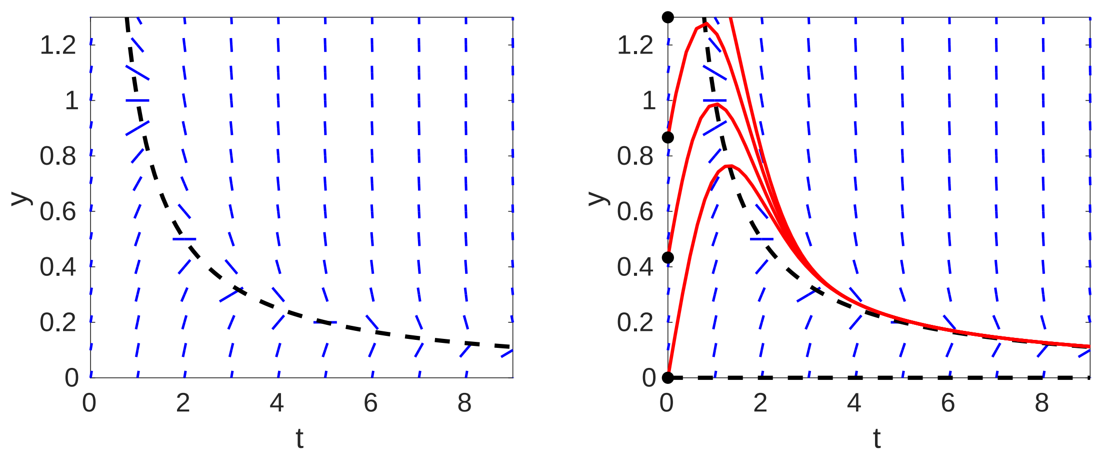
The direction field is shown below. There are no equilibrium points but all solutions approach

$$\ell(t) = 1/t$$

as $t \to \infty$. The direction field suggests this is true since

$$y' > 0 \text{ if } y < \ell(t), \quad y' < 0 \text{ if } y > \ell(t).$$

Intuitively, the ODE wants to push solutions towards the line $y = 1/t$. Note that $\ell(t)$ is **not a solution** to the ODE, unlike the equilibrium points of the first example.



(Note: the direction field arrows are horizontal on the line $y = 1/t$; some of the arrows that are not flat intersect $y = 1/t$ because they've been drawn large).

Remark: It takes more effort to show that

$$y(t) = \frac{1}{t} + \text{smaller terms as } t \to \infty.$$

This can be done via **asymptotics** (not covered here), or by finding the exact solution and doing a bit of calculus.

3 Autonomous equations

Now we shift perspective and study the qualitative behavior of ODEs (without finding exact solutions). Our goal is to describe the dynamics of *autonomous equations*:

$$y' = f(y),$$

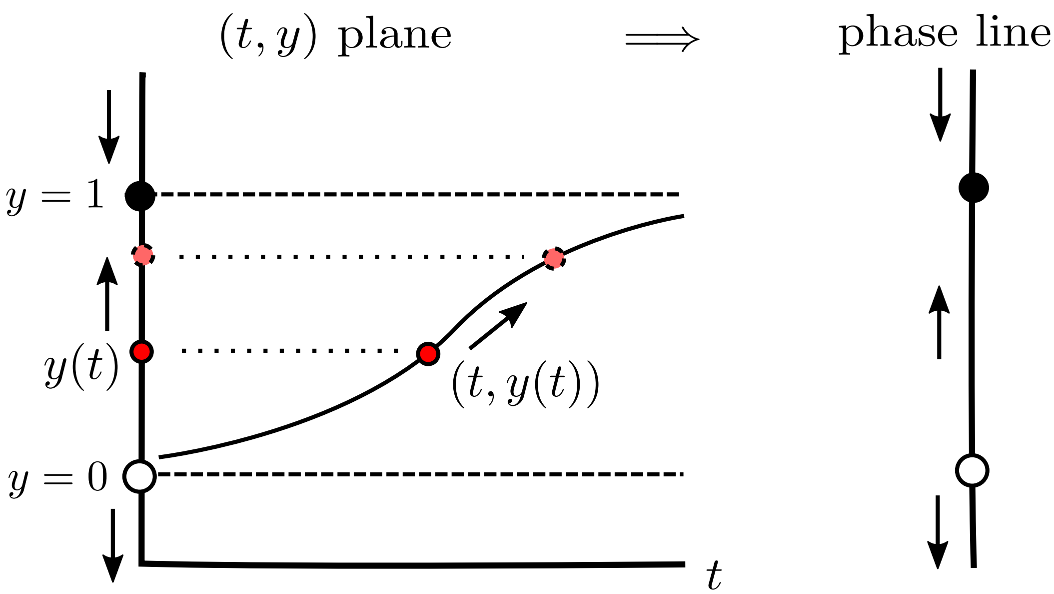
where the independent variable does not appear explicitly. The main questions are

- What is the long term behavior of solutions?
- How does this structure depend on the ODE function f?

We could solve the equation exactly (it is separable), but this is not the most useful approach! Instead, first observe that if y_1 and y_2 both solve the ODE with

$$y_1(t_1) = a, \qquad y_2(t_2) = a$$

then y_2 is the same as y_1 up to a shift in time (due to uniqueness). This means we can project solutions onto the y-axis without ambiguity. In this way, $y' = f(y)$ describes *one-dimensional* flow on a line (see below; the purple dots show solutions moving as time increases).



We represent these dynamics using a **phase line** (the y-axis from the plot above), which provides information about **where solutions go** as t increases or decreases.

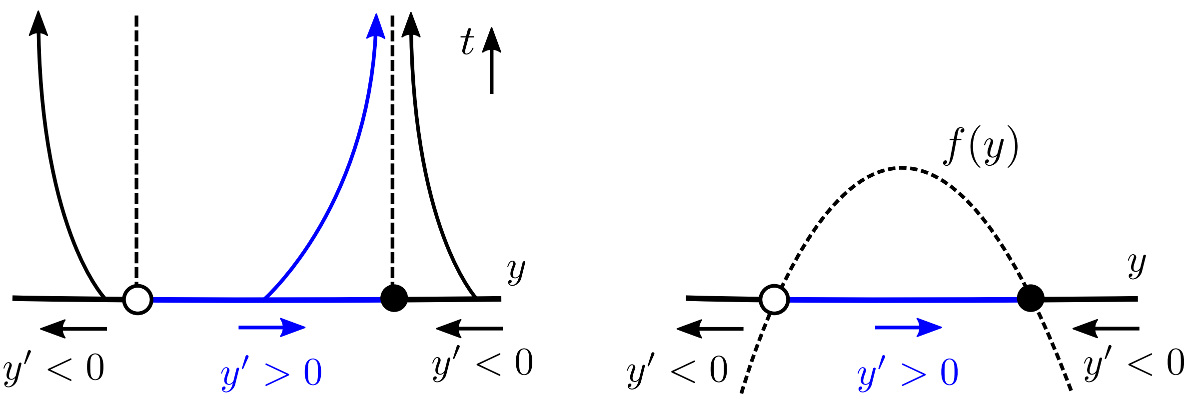
Next, we see how to efficiently draw phase lines and interpret them.

3.1 Using phase lines

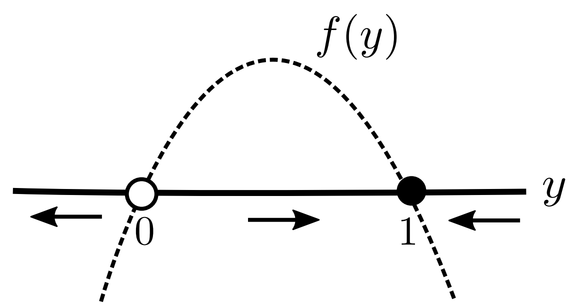
The phase line is usually drawn horizontally. The phase line for

$$y' = y(1-y)$$

is shown below, along with the associated plot in the (t, y) plane (but tilted 90 degrees).



As a visual aid, it is standard to plot y' over the phase line. We now have a representation of how solutions to this ODE behave with a single line:



The diagram shows that at a point y_0 on the line,

- If $y' > 0$ then $y(t)$ increases (move to the right)
- If $y' < 0$ then $y(t)$ decreases (move to the left)
- If $y' = 0$ then $y(t)$ stay constant.

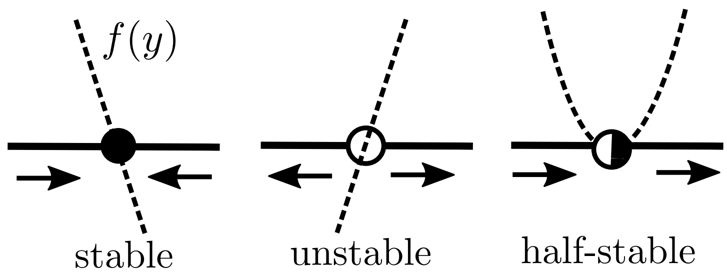
As we just discussed, it is reasonable to refer to 'the solution' $y(t)$ through y_0 because the initial time t_0 does not matter (only true for autonomous equations!).

3.2 Equilibria and stability

A point y^* where $f(y^*) = 0$ is called an **equilibrium point**. These are exactly the points where the constant solution $y(t) \equiv y^*$ solves the ODE. An equilibrium point is

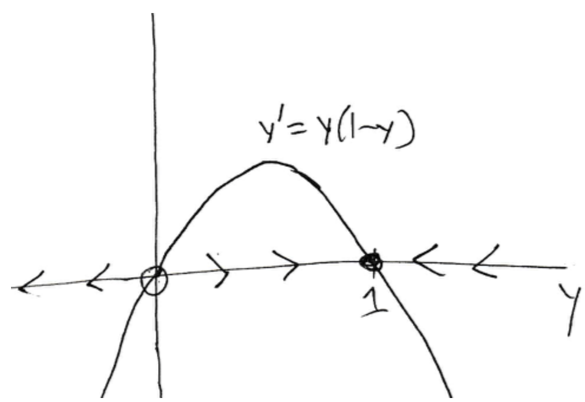
- **(asymptotically) stable** if all solutions near y^* converge to y^*,
- **unstable** if solutions near y^* move away from y^*,
- **half-stable** if it is stable on one side and unstable on the other.

Locally, the phase lines at each type of equilibria look like:



The pictures above (plus one more half-stable case) are all the possibilities, since the stability depends only on the sign of y'. This means that the phase line gives a **complete description of** equilibria, stability, and limits of solutions!

To complete the original example, our phase line for $y' = y(1 - y)$, fully marked[4], is



Now suppose a solution $y(t)$ starts at a value y_0. From the phase line, we can conclude that

- If $y_0 \in (1, \infty)$ then y decreases to 1
- If $y_0 \in (0, 1)$ then y increases to 1
- If $y_0 \in (-\infty, 0)$ then y diverges to $-\infty$.

[4]Stable/unstable equilibria are denoted by filled in and hollow circles; half-stable equilibria are filled in on the stable side

Note that the phase line *does not* tell us how fast solutions converge/diverge (in fact, the divergence for $y_0 < 0$ takes place in finite time). For that, we would need another method, e.g. solving the ODE exactly. The phase line analysis can be applied to any autonomous ODE

$$y' = f(y)$$

and shows that all solutions of are monotonic (why?) and must do one of three things as $t \to \infty$ (or $t \to -\infty$, which one obtains by reversing directions on the phase line):

i) remain constant for all time,

i) converge to an equilibrium point, or

ii) diverge to $\pm\infty$,

and the behavior for any given y_0 can be read from the phase line. We have therefore obtained a full qualitative description of flows on the line.

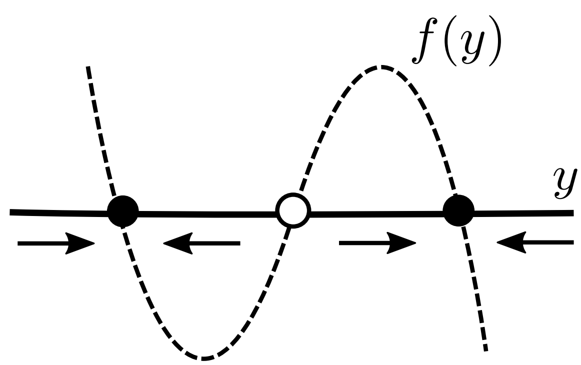
Example: Consider

$$y' = y - y^3.$$

Given an inital value y_0, we can use the phase line to determine

$$\lim_{t\to\infty} y(t).$$

The equilibria are at 0 and ± 1, and the phase line is



which we can deduce from sketching $y - y^3$ (solutions are increasing where this is positive).

The directions point towards 1 when $y_0 > 0$ and towards -1 when $y_0 < 0$. Thus,

- If y_0 is negative, then oslutions converge to -1 as $t \to \infty$
- If y_0 is positive, solutions converge to 1 as $t \to \infty$

This system is an example of a **bistable system**, which has two stable equilibria, but which one you end up in depends on where you start.

3.3 Modeling, bifurcations and fish

Things become more interesting when we add parameters. As a parameter changes, the structure of the equilibrium points may change - equilibria may appear or disappear, drastically changing the phase line. We will not explore this in detail (this is the subject of **dynamical systems** and non-linear ODEs).

Model

To show the idea, we consider modeling a populationl of fish being harvested from a lake. We are interested in knowing how much can be harvested sustainably.

Let $P(t)$ be the population of fish. We make some assumptions to build the model:

i) In the absence of other effect,s the fish grow at a rate rP

ii) A maximum of K fish can live in the lake (the 'capacity'), which we model by multiplying the growth rate by a factor $(1 - P/K)$.

iii) Fish are harvested at a constant rate H.

The model equation is

$$\frac{dP}{dt} = rP(1 - P/K) - H. \tag{4}$$

A harvest rate is **sustainable** if the model predicts that the ODE has a **stable** equilibrium at some non-zero value.

Thus, in physical and mathematical terms, the question to answer is:

- Physical: what is the largest harvest rate such that the fishing is sustainable?
- Math: what is the largest H such that (4) has a stable, positive equilibrium?

Now we can answer the question using the model - and the phase line is exactly what is needed to inspect the possible equilibria.

Analysis

The phase line structure depends on H.

To start, note that for $H = 0$, we have

$$\frac{dP}{dt} = rP(1 - P/K) := f(P)$$

the logistic equation from before. There are two equilibria at

$$P_1 = 0, \qquad P_2 = K.$$

Note that the ODE function $f(P)$ has a maximum at $rK/4$.

Now as H is increased, the phase line diagram changes in that **the graph of $f(P)$ slides downward**, which moves the equilibria (where it intersects the phase line).

- P_1 increases towards $K/2$ (as H increases)
- P_2 decreases towards $K/2$ (as H increases)
- Once H has moved down by more than $rK/4$, both equilibria disappear

Thus, there are three cases to identify. Phase lines are sketched on the next page.

A) Two equilibria: If $0 < H < rK/4$, then there are two equilibria with

$$0 < P_1 < \frac{K}{2} < P_2 < K.$$

The larger one is stable. The model says that the harvest rate will be sustainable, so long as the population does not fall below P_1. If it does, then all the fish will die out.

B) One equilibrium: If $H = rK/4$ exactly, there is one half-stable equilibrium at $K/2$. Both P_1 and P_2 have merged together, changing the phase line structure (this is called a **bifurcation**).

Technically, the fishing is sustainable if the population is above $K/2$. However, if there is any decrease - like a bit of overfishing, or a disease that reduces the population - then the fish will die out.

C) No equilbria: If $H > rK/4$ there are no equilibria at all. The population just decreases and the fish die out.

Conclusions: Returning to the original question, we see that the harvest can only be sustainable when

$$H < H_c \text{ where } H_c := rK/4$$

Thus, to maximize the harvest, H should be chosen close to the 'critical' value H_c. How close depends on how large a population interval one wants for stability.

Phase lines as H increases: (Here $H_c = rK/4$ is the critical harvest value)