Math 353 Lecture Notes
More topics on PDEs
Self-adjoint operators and projection

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1 Preliminaries: self-adjoint operators

In studying the heat equation, we encountered the operator

$$L = -\frac{d^2}{dx^2}$$

along with boundary conditions like

$$\phi(0) = \phi(1) = 0$$

in calculating the eigenfunctions. Without delving too much into the theory (**Sturm-Liouville theory**), it is useful to identify the property that allows the eigenfunction method to work.

For convenience, let us consider the operator

$$L = -\frac{d^2}{dx^2}, \qquad \text{for } L^2 \text{ functions defined on } [a,b] \tag{1.1}$$

and the Dirichlet boundary conditions

$$\phi(a) = \phi(b) = 0. \tag{1.2}$$

When solving eigenvalue problems, etc. with these boundary conditions, we look in the space

$$S = \{v \in L^2[a,b] \text{ such that } v(a) = v(b) = 0\}$$

i.e. functions that satisfy the boundary conditions. Recall also that the inner product

$$\langle v, w\rangle = \int_a^b v(x)w(x)\,dx$$

is well defined for such functions (this is the L^2 part).

Now, a calculation. Let $u(x)$ and $v(x)$ be **any** functions in S. Then

$$\begin{aligned}\langle Lu, v\rangle &= \int_a^b -u''(x)v(x)\,dx \\ &= -u'v\Big|_a^b + \int_a^b u'v'\,dx \\ &= (uv' - u'v)\Big|_a^b - \int_a^b uv''\,dx \\ &= (uv' - u'v)\Big|_a^b + \langle u, Lv\rangle\end{aligned}$$

This is **Green's formula** for the operator L.

But u and v both satisfy the boundary conditions (1.2). The 'boundary terms' from integration by parts vanish, leaving

$$\langle Lu, v\rangle = \langle u, Lv\rangle \text{ for all } u, v \text{ satisfying the BCs.} \tag{1.3}$$

This important property deserves a box:

Self-adjointness: Given an operator L and some boundary conditions, we say that L with those BCs is **self-adjoinht** if

$$\langle Lu, v\rangle = \langle u, Lv\rangle \text{ for all } u, v \text{ that satisfy the BCs.}$$

This property involves three parts: an **interval** $[a, b]$, an **operator** L, and **boundary conditions**.

Note that for $L = -d^2/dx^2$ in particular we also have **Green's formula**

$$\langle Lu, v\rangle = (uv' - u'v)\Big|_a^b + \langle u, Lv\rangle \tag{1.4}$$

which holds for any L^2 functions u and v.

The nice properties of eigenfunctions/values turns out to be guaranteed so long as the operator L is self-adjoint (plus a few technical conditions) - this is the basis of **Sturm-Liouville theory**.

The theory allows one to extend the eigenfunction method to more operators, in particular those of the form

$$Lu = -(p(x)u_x)_x + q(x)u, \qquad p(x) > 0 \text{ in } [a, b].$$

1.1 Examples (self-adjoint operators):

Example 1 (not self-adjoint: Consider

$$L\phi = \phi'$$

in $[a,b]$ with any boundary conditions. Then

$$\langle Lu, v\rangle = \int_a^b u'v\,dx = uv\Big|_a^b - \int_a^b uv'\,dx = (\text{bdry terms}) + \langle u, -v'\rangle$$

There is no hope of L being self-adjoint since clearly

$$\langle u, Lv\rangle = \langle u, v'\rangle \neq \langle u, -v'\rangle.$$

Integration by parts once gives a minus sign that makes the property fail.
Compare to $L\phi = \phi''$, where IBP twice has the minus signs cancel (and it is self-adjoint with the right BCs): $\langle u'', v\rangle = (\text{bdry terms}) + \langle u, v''\rangle$.

Example 2 (not self-adjoint): We check self-adjointness for the boundary value problem

$$xy'' = \lambda y, \quad y(0) = 0, \quad y'(1) = 0.$$

Let $L[y] = xy''$. Then

$$\int_0^1 L[u]v\,dx = \int_0^1 xu''v\,dx = xu'v\Big|_0^1 - \int_0^1 u'(xv)'\,dx.$$

The boundary term vanishes; integrating by parts again we get

$$\int_0^1 L[u]v\,dx = -u(xv)'\Big|_0^1 + \int_0^1 u(xv)''\,dx = -u(1)v(1) + \int_0^1 u(xv)''\,dx.$$

The operator is not self-adjoint (there is a boundary term left, and the integral $\int_0^1 u(xv)''\,dx$ is not $\int_0^1 uL[v]\,dx$, so it fails on two counts).

2 Inhomogeneous boundary conditions

First consider an IBVP with homogeneous BCs:

$$\begin{aligned} u_t &= -Lu + h(x,t), \quad x \in [0,\pi], \;\; t > 0 \\ u(0,t) = 0, &\quad u(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= f(x) \end{aligned} \tag{2.1}$$

with $Lu = -u_{xx}$, which has eigenfunctions $\phi_n = \sin nx$ and $\lambda_n = n^2$ for $n \geq 1$.

Note that since u and ϕ_n satisfy the homogeenous BCs, the self-adjoint property says

$$\langle Lu, \phi_n \rangle = \langle u, L\phi_n \rangle$$

We'll use this to solve the problem. The solution has the form

$$u(x,t) = \sum_{n \geq 1} c_n(t)\phi_n(x), \quad c_n(t) = \frac{\langle u, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{1}{k_n}\langle u, \phi_n \rangle$$

for the usual inner product and $k_n = \langle \phi_n, \phi_n \rangle$ and

$$h(x,t) = \sum_{n \geq 1} h_n(t)\phi_n(x).$$

To find $\langle u, \phi_n \rangle$, we must use the PDE. Take the projection

$$\cdot \to \frac{\langle \cdot, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

of the PDE to get

$$\begin{aligned} \frac{1}{k_n}\langle u_t, \phi_n \rangle &= -\frac{1}{k_n}\langle Lu, \phi_n \rangle + \frac{1}{k_n}\langle h(x,t), \phi_n \rangle \\ c_n'(t) &= -\frac{1}{k_n}\langle Lu, \phi_n \rangle + h_n(t) \end{aligned}$$

since the t-derivative can be swapped with the series sum for u. Now observe that both u and ϕ_n satisfy the homogeneous BCs for L, which is self-adjoint, so

$$\langle Lu, \phi_n \rangle = \langle u, L\phi_n \rangle = \lambda_n \langle u, \phi_n \rangle = \lambda_n k_n c_n(t).$$

It follows that

$$c_n'(t) = -\lambda_n c_n(t) + h_n(t)$$

which is what you would get using the 'plug in the series' method.

Similarly, we can project the ICs (this is the same as in previous examples) to get

$$c_n(0) = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

2.1 inhomogeneous BCs

The value of the projection method is that **it works when u does not have homogeneous BCs**. Let's return to the example above, but now suppose

$$\begin{aligned} u_t &= -Lu + h(x,t), \quad x \in [0,\pi], \;\; t > 0 \\ u(0,t) = e^{-t}, &\quad u(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= f(x) \end{aligned} \tag{2.2}$$

As we will see, the correct eigenvalue problem uses the **homogeneous BCs** (they have to be homogeneous to get eigenfunctions!). The eigenfunctions/values to use are thus the same:

$$\phi_n = \sin nx, \quad \lambda_n = n^2, \quad n \geq 1.$$

Now let u be the solution to the full **inhomogeneous problem**. The key point is that since u does not have homogeneous BCs,

$$\langle Lu, \phi_n \rangle = (\text{bdry terms}) + \langle u, L\phi_n \rangle.$$

The ϕ's are still a basis for functions on $[0,\pi]$ (regardless of the BCs), so

$$u(x,t) = \sum_{n \geq 1} c_n(t)\phi_n(x)$$

Now project the PDE onto ϕ_n, i.e. take

$$\cdot \to \frac{\langle \cdot, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

with $k_n = \langle \phi_n, \phi_n \rangle$ to get

$$c_n'(t) = -\frac{1}{k_n}\langle Lu, \phi_n \rangle + h_n(t).$$

Now write out the boundary values for u and ϕ_n carefully:

$$\phi_n(0) = \phi_n(\pi) = 0, \qquad u(0,t) = e^{-t}, \; u(\pi,t) = 0.$$

By Green's formula, the self-adjoint property almost holds, but **there is a boundary term:**

$$\begin{aligned} \langle Lu, \phi_n \rangle &= -\int_0^\pi u_{xx}\phi_n \, dx \\ &= (u\phi_n' - u_x\phi_n)\Big|_0^\pi + \int_0^\pi u\phi_n'' \, dx \\ &= (u\phi_n' - u_x\phi_n)\Big|_0^\pi + \langle u, L\phi_n \rangle \\ &= u(0,t)\phi_n'(0) + \langle u, L\phi_n \rangle \\ &= ne^{-t} + k_n\lambda_n c_n(t) \end{aligned}$$

since $\phi_n' = n\cos nx$. Thus

$$c_n'(t) = \frac{n}{k_n}e^{-t} + \lambda_n c_n(t) + h_n(t).$$

This is an ODE we can solve - the extra term accounts for the **inhomogeneous BC**. The fact that the ϕ's are an orthogonal basis means all steps here are justified, so $u(x,t)$ with the coefficients we solve for is really the solution.

Why these ϕ's? Note that the projection could be done with any eigenfunction basis to get the projected PDE. For instance, take the example above with no source,

$$u_t = u_{xx}, \quad u(0,t) = e^{-t}, \quad u(\pi,t) = 0$$

but try to use $\phi_n = \cos nx$ instead and

$$u(x,t) = \sum a_n(t)\cos nx.$$

We still have that $L\phi_n = \lambda_n\tilde{\phi}_n$ (and in fact the λ's are the same. This gives

$$a_n'(t) = -\frac{1}{k_n}\langle Lu, \phi_n\rangle$$

$$\langle Lu, \phi_n\rangle = (u_x\phi_n - u\phi_n')\Big|_0^\pi + \langle u, L\phi_n\rangle$$

$$\langle Lu, \phi_n\rangle = u_x\phi_n\Big|_0^\pi - e^{-t}\phi_n'(0) + k_n\lambda_n a_n(t).$$

leading to the nasty equation

$$a_n'(t) = -\lambda_n a_n(t) + (-1)^n u_x(\pi,t) - u_x(0,t)$$

But this is both wrong (the e^{-t} vanishes) and not useful: the u_x terms are unknown since u_x is not given at the boundaries (wrong BCs). We **must have** ϕ_n vanish at the boundaries to cancel out this term - exactly the homogeneous BCs for the problem.

2.2 Example 1

We solve the IBVP

$$\begin{aligned} u_t &= u_{xx},\ x \in (0,\pi),\ t > 0 \\ u_x(0) &= A,\ u_x(\pi) = 0 \\ u(x,0) &= T_0 \end{aligned} \tag{2.3}$$

which describes heat in a metal rod insulated at one end and with a constant output flux A at the other (assuming $A > 0$). The operator is $Lu = -u_{xx}$ and the eigenvalue problem is

$$-\phi'' = \lambda\phi,\ \phi'(0) = \phi'(\pi) = 0 \implies \phi_n = \cos nx,\ \lambda_n = n^2,\ n = 0, 1, 2, \cdots$$

You could try to look for a steady state first:

$$0 = w''(x), \quad w'(0) = A, \quad w'(\pi) = 0$$

so $w(x) = ax + b$. But the BCs then require both $a = A$ and $a = 0$. The failure makes some sense here, because the rod is insulated except that heat is **rmoved**, so it should just keep draining and not reach an equilibrium.

PDE solution: Let u be the solution to the IBVP. Then

$$u(x,t) = \sum_{n=0}^{\infty} c_n(t)\phi_n(x).$$

Now take the inner product of the DE with ϕ_n to get (with $k_n = \langle \phi_n, \phi_n \rangle$)

$$\begin{aligned} c_n'(t) &= -\frac{\langle Lu, \phi_n \rangle}{k_n} \\ &= \frac{1}{k_n}\left(\phi_n u_x - \phi_n' u\right)\Big|_0^\pi - \frac{1}{k_n}\langle u, L\phi_n \rangle. \end{aligned}$$

By the boundary conditions,

$$\begin{aligned} \phi_n'(0) &= \phi_n'(\pi) = 0, \\ u_x(0) &= A,\ u_x(\pi) = 0. \end{aligned}$$

Plugging this into the boundary terms, we get

$$c_n'(t) = -\frac{A\phi_n(0)}{k_n} - \lambda_n c_n.$$

Now from the IC,

$$f(x) = \sum_{n=0}^{\infty} c_n(0)\phi_n(x) \implies c_n(0) = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

But $f = T_0\phi_0$ so it follows that

$$c_0(0) = T_0,\ c_n(0) = 0 \text{ for } n > 0. \tag{2.4}$$

This gives the IVP for the c_n's:

$$c_n' + \lambda_n c_n = -A/\langle \phi_n, \phi_n \rangle, \qquad c_n(0) \text{ given by (2.4)}.$$

Solve the coeff. ODEs: There are two cases. When $\lambda_n \neq 0$,

$$c_n(t) = -\frac{2A}{\pi \lambda_n}(1 - e^{-\lambda_n t})$$

noting that $\langle \phi_n, \phi_n \rangle = \pi/2$ for $n \geq 1$.

But for $\lambda_0 = 0$, we have $c_n(0) = T_0$ and (note that $\langle \phi_0, \phi_0 \rangle = \pi$)

$$c_0' = -A/\langle \phi_0, \phi_0 \rangle \implies c_0(t) = T_0 - \frac{A}{\pi}t.$$

Summarize: Thus, the solution is

$$u(x,t) = T_0 - \frac{A}{\pi}t - \frac{2A}{\pi}\sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n t})}{\lambda_n}\phi_n$$

with $\lambda_n = n^2$ and $\phi_n = \cos nx$ and $\langle \phi_n, \phi_n \rangle = \int_0^\pi \cos^2 nx \, dx$ (you could simplify more). Note that $\langle \phi_0, \phi_0 \rangle = \pi$ and $\langle \phi_n, \phi_n \rangle = \pi/2$; the integrals are different for $n = 0$ and $n \neq 0$.

2.3 Example 2 (lengthy)

A fully worked example similar to the one in Section 2.1. We solve the heat equation in $[0, \pi]$ with a time-dependent boundary condition:

$$\begin{aligned} u_t = u_{xx}, \quad x \in (0,\pi), \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = At, \quad t > 0, \\ u(x,0) = f(x). \end{aligned} \tag{2.5}$$

The eigenfunctions/values are

$$\phi_n = \sin nx, \quad \lambda_n = n^2, \qquad n \geq 1.$$

Write the solution u in terms of the eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x).$$

Now we project the PDE

$$u_t = -Lu$$

onto the eigenfunction ϕ_n using

$$\cdot \to \frac{\langle \cdot, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

with $k_n = \langle \phi_n, \phi_n \rangle$ to get

$$c_n'(t) = -\frac{1}{k_n}\langle Lu, \phi_n \rangle$$

Integrating by parts and/or using Green's formula,

$$c_n'(t) = -\frac{1}{k_n}\left((u\phi_n' - u_x\phi_n)\Big|_0^\pi - \langle u, L\phi_n \rangle\right)$$

and noting that only one of the boundary terms (at $x = \pi$) remains, we get

$$c_n'(t) = -\frac{1}{k_n}u(\pi,t)\phi_n'(\pi) - \lambda_n c_n(t)$$

$$c_n'(t) = -\frac{Ant}{k_n}\cos n\pi - \lambda_n c_n(t)$$

For brevity (note that $k_n = \pi/2$), set

$$\gamma_n = -\frac{2An\cos(n\pi)}{\pi}. \tag{2.6}$$

The ODE for c_n is then

$$c_n'(t) + \lambda_n c_n(t) = \gamma_n t$$

As before, write the initial condition in terms of the eigenfunction basis:

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \tag{2.7}$$

Then $u(x,0) = f(x)$ gives the initial condition for c_n:

$$c_n(0) = a_n.$$

At this point, we are "done" in the sense that the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

where the c_n's are the solutions to the IVPs

$$c_n'(t) + \lambda_n c_n(t) = \gamma_n t, \quad c_n(0) = a_n$$

with

$$\gamma_n = -\frac{2An\cos n\pi}{\pi}, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\phi_n(x)\, dx$$

and $\lambda_n = n^2$ and $\phi_n = \sin nx$. This completely defines the solution.

However, to be complete, we solve the ODEs. Use an integrating factor:

$$(e^{\lambda_n t} c_n)' = \gamma_n e^{\lambda_n t} t$$

to obtain

$$c_n = a_n e^{-\lambda_n t} + \gamma_n e^{-\lambda_n t} \int_0^t e^{\lambda_n s} s \, ds.$$

Evaluating the integral we get

$$c_n(t) = a_n e^{-\lambda_n t} + \frac{\gamma_n}{\lambda_n^2}\left(\lambda_n t - 1 + e^{-\lambda_n t}\right).$$

We can plug in λ_n^2 and γ_n from (2.6) we get

$$c_n(t) = a_n e^{-n^2 t} - \frac{2A\cos(n\pi)}{\pi n^3}\left(n^2 t - 1 + e^{-n^2 t}\right). \tag{2.8}$$

The solution is then given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

with $c_n(t)$ given by (2.8) and the a_n's by (2.7). Note that the first term in the expression (2.8) for $c_n(t)$ gives the solution if the boundary conditions were homogeneous; the second term is the response to the inhomogeneous boundary conditions.