Math 353 Lecture Notes
Intro to PDEs
Laplace's equation in a disk

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Topics covered

- Laplace's equation in a disk
  - Solution (separation of variables)
  - Semi-circles (sections) and annuli
  - Review: Cauchy-Euler equations

1 Laplace's equation in a disk

Separation of variables can be used in geometries other than an interval/rectangle. To do so, we need to have variables such that the boundaries are **separated** - only one variable varies on each (e.g. only x and only y for the rectangle).

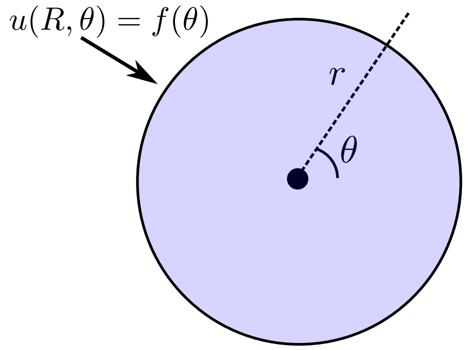
The (x, y) coordinates cannot be used for a disk, but polar coordinates work. Laplace's equation for $u(r, \theta)$ in a disk with a prescribed value $f(\theta)$ on the boundary is

$$
\begin{gathered}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r \in (0, R), \quad \theta \in [0, 2\pi] \\
u(R, \theta) = f(\theta), \quad \theta \in [0, 2\pi]
\end{gathered}
$$

We also need **periodic boundary conditions** in θ and a boundedness condition:

$$u(r, 0) = u(r, 2\pi),\ \ u_r(r, 0) = u_r(0, 2\pi) \tag{1.1}$$

$$u(r, \theta) \text{is bounded for } r \in [0, R] \tag{1.2}$$



Part I (eigenfunctions): The correct way to write the problem in operator terms is

$$u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}Lu = 0, \quad Lu = -u_{\theta\theta}.$$

This is not obvious! To 'derive it', **we can use separation of variables.** Look for solutions

$$u = g(r)h(\theta).$$

Substituting into the PDE we get

$$g''(r)h(\theta) + \frac{1}{r}g'(r)h(\theta) + \frac{1}{r^2}g(r)h''(\theta) = 0$$

$$\implies \frac{r^2g''(r) + rg'(r)}{g(r)} = -\lambda\frac{h''(\theta)}{h(\theta)} \tag{1.3}$$

With the periodic boundary conditions (1.1), we get a familiar eigenvalue problem:

$$h''(\theta) + \lambda h(\theta) = 0, \qquad h(0) = h(2\pi), \quad h'(0) = h'(2\pi)$$

$$\implies h_0 = a_0, \lambda_0 = 0, \quad h_n(\theta) = \cos n\theta \text{ and } \sin n\theta, \quad \lambda_n = n^2, \quad n \geq 1. \tag{1.4}$$

Caution: As a warning, if the PDE is not homogeneous, SoV stops being useful here. At this point, we take the eigenfunctions/values and use the eigenfunction method.

We now solve for g_n from (1.3):

$$r^2g_n''(r) + rg_n'(r) - n^2g_n(r) = 0.$$

The ODE is a Cauchy-Euler equation with roots $\pm n$ (see review below; section 2); the basis solutions are r^n and r^{-n} so the general solution is

$$g_n = \begin{cases} c_nr^n + d_nr^{-n} & n \geq 1 \\ c_0 + d_0 \ln r & n = 0 \end{cases}$$

By the boundedness condition (1.2), $d_n = 0$ (r^{-n} and $\ln r$ are not finite at $r = 0$) so

$$g_n = c_nr^n, \quad n \geq 0.$$

The separated solutions are then $g_n \cos n\theta$ and $g_n \sin n\theta$, or (grouping by eigenvalue),

$$u_0 = \frac{a_0}{2}, \quad u_n = r^n(a_n \cos n\theta + b_n \sin n\theta), \quad n \geq 1$$

for arbitrary constants a_n and b_n (note that 1/2 chosen to match the Fourier series).

Part II (continuing with SoV): Since the PDE is homogeneous, the solution is a linear combination of the u_n's. With $\phi_0 = 1/2$, $\phi_n = \cos n\theta$ for $n \geq 1$ and $\psi_n = \sin n\theta$,

$$u(r,\theta) = a_0\phi_0 + \sum_{n=1}^{\infty} r^n(a_n\phi_n + b_n\psi_n). \tag{1.5}$$

Now to get the constants, impose the boundary condition at $r = R$:

$$f(\theta) = u(R,\theta) = a_0\phi_0 + \sum_{n=1}^{\infty} R^n(a_n\phi_n + b_n\psi_n)$$

so by the usual calculation for the coefficients (with $\langle f, g\rangle = \int_0^{2\pi} f(\theta)g(\theta)\,d\theta$)

$$\begin{aligned} a_n &= \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \begin{cases} \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta & \text{for } n \geq 1 \\ \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta & \text{for } n = 0 \end{cases}, \\ b_n &= \frac{\langle f, \psi_n\rangle}{\langle \psi_n, \psi_n\rangle} = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta. \end{aligned} \tag{1.6}$$

Note that this is just the Fourier series, but using the interval $[0, 2\pi]$. Using the inner product ensures that we have all the constants etc. right (vs. quoting the Fourier series formula).

The process is now complete; the solution is the Fourier series (1.5) with coefficients (1.6).

Remark: If the problem were inhomogeneous, we would consider

$$u(r,\theta) = a_0(r)\phi_0 + \sum_{n=1}^{\infty} a_n(r)\phi_n + b_n(r)\psi_n$$

then plug this into the PDE and proceed as in the eigenfunction method.

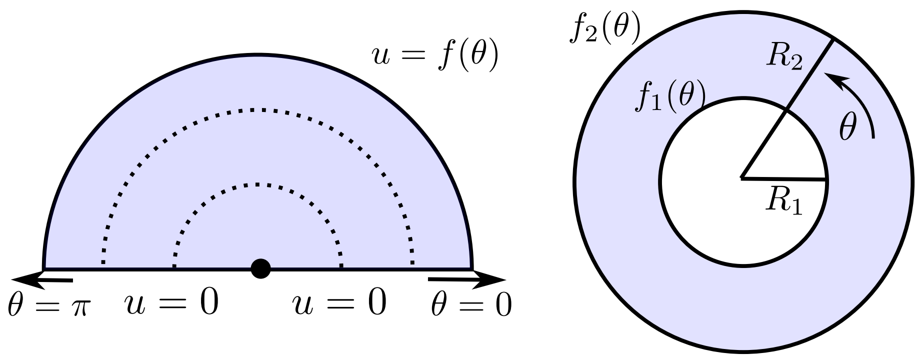
1.1 Other shapes

The same method can be used to solve Laplace's equation (or other PDEs) in any domain where the boundaries are all 'separable', i.e. of the form

$$\text{variable} = \text{const.}$$

e.g. $r = R$ for the circle or $x = 0, B$ and $y = 0, A$ for the rectangle's four sides. Otherwise, other techniques (beyond the scope of the course) must be used. This is required to get eigenfunctions in **one direction** that don't depend on the other (e.g. just $\phi(\theta)$, not $\phi(\theta, r)$).

In polar coordinates, this means that sections ($\Theta_1 \leq \theta \leq \Theta_2$) and annuli ($R_1 \leq r \leq R_2$) and are also allowable domains (sketched below).



Boundary conditions (polar): An annulus and section are slightly different from the circle - some implied boundary conditions become explicit ones.

- In a **section**, there are flat boundaries out of the origin at the θ endpoints.
- In an **annulus**, there are two boundaries in r where BCs can be specified.

For a section, the 'periodic BCs' are replaced by actual BCs at the θ endpoints.
For an annulus, the 'bounded' condition is replaced by the inner boundary.

1.2 Example (semi-circle)

Consider the semi-circle (upper left figure on previous page)

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, &\quad r \in (0,2), \quad \theta \in [0, \pi/2] \\ u(r,0) = u(r,\pi) &\quad r \in (0,R) \\ u(2,\theta) = f(\theta), &\quad \theta \in [0,\pi] \end{aligned}$$

Look for separated solutions $g(r)\phi(\theta)$, leading to

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(\pi) \text{ in } [0,\pi], \qquad g'' + \frac{1}{r}g' - \frac{\lambda}{r^2}g = 0.$$

Note there are actual BCs at $\theta = 0, \pi$. The eigenvalue problem is familiar and has solutions

$$\phi_n = \sin(n\theta), \quad \lambda_n = n^2, \quad n \geq 1.$$

The r equation is the same as for the full circle except that $\lambda \neq 0$, yielding

$$g_n(\theta) = c_n r^n + d_n r^{-n} \implies g_n = c_n r^n \text{ for } n \geq 1.$$

There is no $n = 0$ case to be concerned about. Thus the solution is, by superposition,

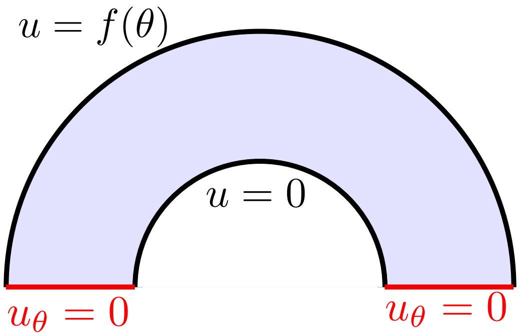
$$u(r,\theta) = \sum_{n\geq 1} c_n r^n \sin n\theta.$$

and the c_n's are determined by the BC at $r = 1$, $u(1,\theta) = f(\theta)$.

1.3 Example (annulus):

Consider the half-annulus with Neumann BCs,

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, &\quad r \in (1,2), \quad \theta \in [0,\pi] \\ \text{(flat end): } u_\theta(r,0) = u_\theta(r,\pi) &\quad r \in (1,2) \\ \text{(inner): } u(1,\theta) = 0, &\quad \theta \in [0,\pi] \\ \text{(outer): } u(2,\theta) = f(\theta), &\quad \theta \in [0,\pi] \end{aligned}$$



The SoV steps work the same as before. The eigenfunctions/values are

$$\phi_n(\theta) = \cos n\theta, \quad \lambda_n = n^2, \quad n \geq 0.$$

However, $\lambda = 0$ is now an eigenvalue. The solution has the form

$$u(r,\theta) = g_0(r)\phi_0(\theta) + \sum_{n\geq 1}(c_n r^n + d_n r^{-n})\phi_n(\theta).$$

As before, we have

$$g_n(r) = \begin{cases} c_n r^n + d_n r^{-n} & n \geq 1 \\ c_0 + d_0 \ln r & n = 0 \end{cases}$$

but neither term is infinite in $[1, 2]$ so they cannot be excluded. The BCs at $r = 1$ and $r = 2$ both must be applied to find the coefficients. Let $\langle f, g\rangle = \int_0^\pi f(\theta)g(\theta)\, d\theta$.
First apply the (inner) BC at $r = 1$:

$$0 = u(1,\theta) \implies g_n(0) = 0$$

$$\implies d_n = -c_n \text{ for } n \geq 1, \quad c_0 = 0.$$

Then apply the (outer) BC at $r = 2$:

$$f(\theta) = u(2,\theta) = d_0 \ln 2\, \phi_0 + \sum_{n\geq 1} c_n(2^n - 2^{-n})\phi_n(\theta)$$

$$\implies c_n = \frac{1}{2^n - 2^{-n}} \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} \text{ for } n \geq 1 \tag{1.7}$$

and for $n = 0$ (recall that $\phi_0 = 1$ here)

$$d_0 \ln 2 = \frac{\langle f, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle} = \frac{1}{\pi}\int_0^\pi f(\theta)\, d\theta. \tag{1.8}$$

In summary, with d_0 and c_n given by (1.8) and (1.7),

$$u(r,\theta) = d_0 \ln r + \sum_{n\geq 1} c_n \left(r^n - \frac{1}{r^n}\right) \cos n\theta$$

2 Review: Cauchy-Euler equations

A type of ODE that can be solved exactly (appearing, previously, on the HW).
A **Cauchy-Euler** or **equidimensional** ODE of second order has the form

$$ax^2y'' + bxy' + cy = 0, \quad x > 0 \text{ or } x < 0. \tag{2.1}$$

To solve, guess a 'trial solution' of the form x-to-a-power. Since

$$y(x) = x^\gamma \implies xy' = \gamma x^\gamma, \quad x^2y'' = \gamma(\gamma-1)x^\gamma$$

we have that x^γ is a solution if and only if

$$p(\gamma) = a\gamma(\gamma-1) + b\gamma + c = 0 \tag{2.2}$$

where $p(\gamma)$ is the 'characteristic polynomial'. We need two linearly independent solutions to solve (2.1). If $\gamma_1 \neq \gamma_2$ are real, we are done (two solutions). Otherwise:

- If γ is a repeated root, then multiply by a factor of $\ln x$ (i.e. solns x^γ and $x^\gamma \ln x$).

- $\gamma = s + \omega i$ is complex, take real/imaginary parts to get two solutions:

$$x^r = e^{(s+\omega i)\ln x} \implies x^s\cos(\omega \ln x),\ x^s \sin(\omega \ln x).$$

- For negative x, replace with $|x|$.

In summary, the solution procedure is:

1) Plug in the trial solution x^γ and find the characteristic polynomial $p(\gamma)$.

2) Calculate the roots γ_1, γ_2 of $p(\gamma)$

3) The solution depends on the roots (three cases):

$$\begin{aligned}
\text{roots } \gamma_1 \neq \gamma_2, \text{ real } &\implies y = c_1|x|^{\gamma_1} + c_2|x|^{\gamma_2} \\
\text{root } \gamma \text{ (repeated) }, &\implies y = c_1|x|^{\gamma} + c_2x^{\gamma}\ln|x| \\
\text{roots } \gamma = s \pm \omega i \text{ (complex) } &\implies y = c_1|x|^s\cos(\omega\ln|x|) + c_2|x|^s\sin(\omega\ln|x|)
\end{aligned}$$

Remark: The cases are 'like a LCC equation, but with $\ln x$ instead of x' and the characteristic polynomial is (2.2) instead of $a\gamma^2 + b\gamma + c$ (for LCC). In fact, one can convert the Cauchy-Euler equation into an LCC one by using $\ln x = t$.