Math 353 Lecture Notes
Steady states and Laplace's equation

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Topics covered

- Steady states
  - Inhomogeneous BCs (time-independent)
  - Reducing to homogeneous case with a 'steady state'
- Laplace's equation
  - Definition; connection to heat equation
  - Solution in a rectangle/square
  - Use of the right basis function (sinh, translation)
  - Eigenfunction vs. coefficient direction
  - Superposition tricks

1 Steady states

The following both (a) a way to describe the limit as $t \to \infty$ for the heat equation and (b) a way to convert inhomogeneous problems into (easier) homogeneous ones.

Consider the IBVP

$$\begin{gathered} u_t = u_{xx} + h(x), \quad x \in [0, \ell], \;\; t > 0 \\ u(0,t) = A, \quad u(\ell, t) = B, \quad t > 0 \\ u(x,0) = f(x) \end{gathered} \tag{1.1}$$

which describes heat flow with a time independent source $h(x)$ and temperature fixed at both ends at different values A and B. Over time, the heat will diffuse and approach a 'steady state' (equilibrium):

$$\overline{u}(x) = \lim_{t \to \infty} u(x,t).$$

The key point is that **the steady state is a solution to the PDE + BCs that does not depend on time**. Taking $t \to \infty$, in the PDE, the u_t term vanishes, leaving

$$\begin{gathered} 0 = u_{xx} + h(x), \quad x \in [0, \ell] \\ u(0) = A, \quad u(\ell) = B \end{gathered}$$

This is an ODE for $\overline{u}(x)$ that can be easily solved before dealing with the PDE, which suggests that it is a good way to handle the inhomogeneous terms.

Solution procedure: To start, look for a time-independent solution $w(x)$ (not yet known to be the steady state) to the PDE + BCs:

$$\begin{gathered} 0 = w_{xx} + h(x), \quad x \in [0, \ell] \\ w(0) = A, \quad w(\ell) = B \end{gathered} \tag{1.2}$$

Solve this ODE boundary value problem to get $w(x)$ and then examine the difference

$$v(x,t) = u(x,t) - w(x).$$

The function v, by superposition, solves the **homogeneous** IBVP

$$\begin{gathered} v_t = v_{xx}, \quad x \in [0, \ell], \ \ t > 0 \\ v(0,t) = 0, \quad v(\ell, t) = 0, \quad t > 0 \\ v(x,0) = f(x) - \overline{u}(x) \end{gathered} \tag{1.3}$$

To see this, take the difference of each equation (PDE, BCs and IC) for u and $\overline{u}$:

$$\begin{cases} u_t = u_{xx} + h(x) \\ 0 = \overline{u}_{xx} + h(x) \end{cases} \implies v_t = v_{xx},$$

$$\begin{cases} u(0,t) = A \\ \overline{u}(1) = A \end{cases} \implies v(0,t) = 0$$

with the same for the BC at $x = \ell$ and the initial conditions. Now (1.3) is the 'easy case' (homogeneous PDE and BCs). The solution to the original IBVP (1.1) is then

$$u(x,t) = w(x) + v(x,t).$$

Analysis (steady state): Now suppose we wish to show that $w(x)$ is the steady state for the problem. We observed that, by taking $t \to \infty$ in the IBVP,

$$\text{if a steady state } \overline{u} = \lim_{t\to\infty} \text{ exists, it must be } w(x).$$

To verify that $\overline{u}$ is the computed function $w(x)$, it suffices to show that

$$\lim_{t\to\infty} v(x,t) = 0.$$

Since v will have the form $\sum_n c_n(t)\phi_n(x)$, the limit can be shown by verifying that $c_n(t) \to 0$ for all n, which often means showing that

$$\lambda_n > 0 \text{ for all } n.$$

Note that the steady state can be computed **without solving the full problem.**

Procedure (steady state): To solve problems for u with inhomogeneous terms that are **time-independent**,

- Find a time-independent solution w to the PDE + BCs
- Write the problem for the difference $v = u - \overline{u}$
- Solve this (homogeneous) problem

If a steady state exists, it must be $\overline{u}$ (as solved above). To show that it is the limit, show that $v(x,t) \to 0$ as $t \to \infty$ (e.g. show $\lambda_n > 0$ for all n).

Even if $\overline{u}$ is not the limit, the procedure works, but then $\overline{u}$ has less significance.

1.1 Standard example

A steady state is used to solve the inhomogeneous problem

$$\begin{gathered} u_t = u_{xx} - 6x, \quad x \in (0,1), \ \ t > 0 \\ u(0,t) = 0, \quad u(1,t) = 3, \quad t > 0 \\ u(x,0) = f(x). \end{gathered}$$

We show that u converges to the steady state as $t \to \infty$. This is done in two parts:

Part 1: Find a time independent solution: Look for a time-independent solution

$$u(x,t) = w(x).$$

Plug into the PDE and BCs (ignore the initial condition) to get

$$w_{xx} = 6x, \qquad w(0) = 0, \ \ w(1) = 3.$$

Integrate the ODE and apply the BCs to get

$$w(x) = x^3 + ax + b \implies w(x) = x^3 + 2x. \tag{1.4}$$

Part 2: Solve for the difference; verify the steady state: Now let

$$v(x,t) = u(x,t) - \overline{u}(x).$$

By the superposition argument, v satisfies the homogeneous IBVP

$$\begin{gathered} v_t = v_{xx}, \quad x \in (0,1), \ \ t > 0 \\ v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0 \\ v(x,0) = f(x) - \overline{u}(x) \end{gathered}$$

This homogeneous problem (Dirichlet BCs) was solved before; the result is

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin \lambda_n x, \tag{1.5}$$

$$\lambda_n = n^2\pi^2, \quad b_n = 2\int_0^1 v(x,0) \sin n\pi x\, dx = 2\int_0^1 (f(x) - \overline{u}(x)) \sin \pi x\, dx.$$

The solution to the IBVP is then $u(x,t) = w(x) + v(x,t)$.

Part 2b (analysis): Since all the eigenvalues are positive, it follows from (1.5) that

$$v(x,t) = \sum (\text{exp. decaying terms}) \implies \lim_{t\to\infty} v(x,t) = 0$$

so we can conclude that $\overline{u}(x)$ is really the steady state:

$$\lim_{t\to\infty} u(x,t) = w(x) + \lim_{t\to\infty} v(x,t) = w(x).$$

Remark (minimal version): Suppose we only wanted to know the steady state and verify that it is th elimit. Then it suffices to solve for $w(x)$ and show the eigenvalues are positive:

$$w(x) = x^3 + 2x, \qquad \lambda_n = n^2\pi^2,\ n \geq 1 \implies \lambda_n > 0$$

The rest of the solution (coefficients b_n etc.) are not needed.

1.2 When does the method fail?

This trick works when there is a steady state but *only when the source term and boundary conditions do not depend on time.* For instance,

$$u_t = t u_{xx} + \sin x$$

cannot be solved using this method. Assuming $u_t = 0$ is not enough since we also need to take $t \to \infty$ and we cannot find a $u = w(x)$ that solves

$$t w''(x) + \sin x = 0.$$

The eigenfunction method must be used to find the steady state directly. Similarly, if

$$u(0,t) = \sin t$$

then the problem for $w(x)$ would have $w(0) = \sin t$ - not possible for a function of x.

1.3 Example (Non-uniqueness)

It is not always true that the first step is enough to find $\overline{u}(x)$. Consider the following problem with Neumann BCs:

$$\begin{aligned} u_t = u_{xx} + x - 1, &\quad x \in (0,2), \;\; t > 0 \\ u_x(0,t) = 0, \quad u_x(2,t) = 0, &\quad t > 0 \\ u(x,0) = f(x). & \end{aligned}$$

Solving for a time-independent solution $w(x)$ (not yet known to be the steady state), we find

$$w_{xx} = x - 1, \quad w'(0) = 0, \quad w'(2) = 0.$$

$$\begin{aligned} \text{ODE} &\implies w = (x-1)^3/6 + ax + b \\ \text{BCs} &\implies w = (x-1)^3/6 - \frac{1}{2}x + b. \end{aligned}$$

Both BCs require only $a = 1/2$; the value of b appears arbitrary. The 'limit of the PDE' procedure gives a set of possible equilibria $w(x)$, one for each $b \in \mathbb{R}$. The steady state $\overline{u}(x)$ is one of these equilibria - but the value of b is influenced by the initial condition.

Complete solution (the hard way): To find it directly, set

$$v = u - w$$

and solve the homogeneous problem for v to get (as before)

$$v(x,t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \cos \frac{n\pi x}{2}, \quad \lambda_n = (n\pi/2)^2, \; n \geq 0.$$

The eigenvalues are non-negative, so the limit as $t \to \infty$ (there is a steady state), but the $n = 0$ term of v survives the limit ($\lambda_0 = 0$). Using the solution to compute $\overline{u}(x)$,

$$\begin{aligned} \overline{u}(x) &= \lim_{t \to \infty} u(x,t) \\ &= w(x) + \frac{1}{2}\int_0^2 f(x)\,dx - \frac{1}{2}\int_0^2 w(x)\,dx \\ &= \left(w(x) - \frac{1}{2}\int_0^2 w(x)\,dx \right) + \frac{1}{2}\int_0^2 f(x)\,dx. \end{aligned}$$

Plugging in $w(x)$, the unknown b cancels out, giving the (unique) steady state

$$\overline{u}(x) = \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1) + \frac{1}{2}\int_0^2 f(x)\,dx.$$

2 Laplace's equation

In two dimensions the heat equation[1] is

$$u_t = \alpha(u_{xx} + u_{yy}) = \alpha \Delta u$$

where $\Delta u = u_{xx} + u_{yy}$ is the **Laplacian** of u (the operator Δ is the 'Laplacian'). If the solution reaches an equilibrium, the resulting steady state will satisfy

$$u_{xx} + u_{yy} = 0. \tag{2.1}$$

This equation is **Laplace's equation** in two dimensions, one of the essential equations in applied mathematics (and the most important for time-independent problems). Note that in general, the Laplacian for a function $u(x_1, \cdots, x_n)$ in $\mathbb{R}^n \to \mathbb{R}$ is defined to be the sum of the second partial derivatives:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Laplace's equation is then compactly written as

$$\Delta u = 0.$$

The inhomogeneous case, i.e.

$$\Delta u = f$$

the equation is called **Poisson's equation**. Note that the steady states of the previous section in 1d, e.g.

$$u_t = u_{xx} \xrightarrow{t \to \infty} 0 = u_{xx}$$

lead to Laplace's equation in 1d (we did not name it as such since it is just an ODE in x; things are more interesting in 2d!).

Innumerable physical systems are described by Laplace's equation or Poisson's equation, beyond steady states for the heat equation: inviscid fluid flow (e.g. flow past an airfoil), stress in a solid, electric fields, wavefunctions (time independence) in quantum mechanics, and more.

The two differences with the wave equation

$$u_{tt} = c^2 u_{xx}$$

are:

- We specify **boundary conditions** in both directions, not initial conditions in t.
- There is an opposite sign; we have $u_{xx} = -u_{yy}$ rather than $u_{tt} = c^2 u_{xx}$.

The first point changes the way the problem is solved slightly; the second point changes the answer. Note that there is also no coefficient, but this is not really important (we can just as easily solve $u_{xx} + k^2 u_{yy} = 0$).

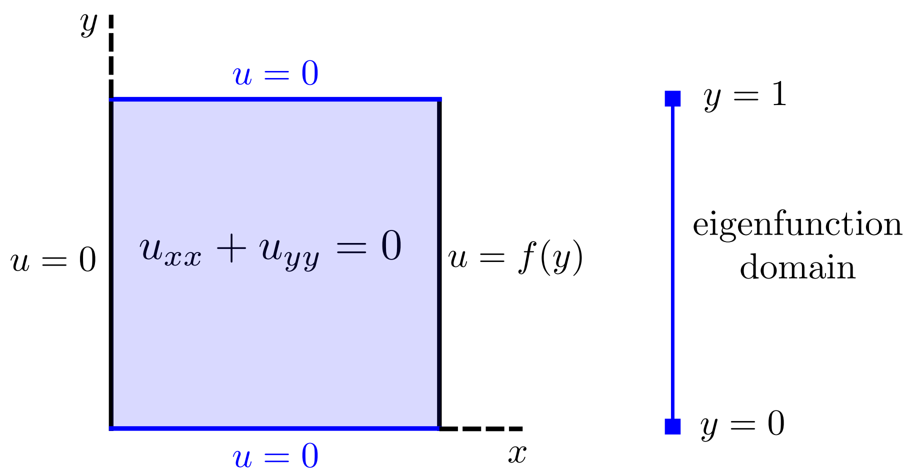
[1] The derivation follows the same argument made in one dimension, but using the divergence theorem instead of the fundamental theorem of calculus.

2.1 Solution in a rectangle

We can solve Laplace's equation in a bounded domain by the same techniques used for the heat and wave equation.

Consider the following boundary value problem in a square of side length 1:

$$\begin{aligned} 0 &= u_{xx} + u_{yy}, \quad x \in (0,1), \quad y \in (0,1) \\ u(x,0) &= 0, \quad u(x,1) = 0, \quad x \in (0,1) \\ u(0,y) &= 0, \quad u(1,y) = f(y), \quad y \in (0,1). \end{aligned}$$



The boundary conditions are all homogeneous (shown in **blue** above) except on the right edge ($y = 0$)). Motivated by this, we will try to get eigenfunctions $\phi(y)$, since the eigenvalue problem requires us to impose **homogeneous boundary conditions**.

Look for a separated solution

$$u = g(x)\phi(y).$$

Substitute into the PDE to get

$$0 = g''(x)\phi(y) + g(x)h''(y)$$

and then separate:

$$-\frac{h''(y)}{\phi(y)} = \frac{g''(x)}{g(x)} = \lambda.$$

This leads to the pair of ODEs

$$\phi''(y) + \lambda\phi(y) = 0, \qquad g''(x) = \lambda g(x).$$

Applying the boundary conditions on the sides $x = 0$ and $x = 1$, we get the BVP

$$\phi''(y) + \lambda\phi(y) = 0, \quad \phi(0) = \phi(1) = 0.$$

We know the solutions to the above; they are

$$\phi_n(y) = \sin n\pi y, \quad \lambda_n = n^2\pi^2, \qquad n \geq 1.$$

Now we solve for g for each λ_n. Note that there is only one boundary condition (at $x = 1$); we leave the $f(x)$ condition for later (it will require using the full series). We solve

$$g'' - n^2\pi^2 g = 0, \qquad g(0) = 0$$

to get

$$g_n(x) = a_n \sinh n\pi x.$$

The solution

$$u_n = g_n(x)\phi_n(y)$$

satisfies the PDE and all the boundary conditions except $u(x, 0) = f(x)$. To satisfy this, we need to write u as a sum all of the separated solutions $g_n\phi_n$:

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh n\pi x \ \ \sin n\pi y.$$

Now apply $u(1, y) = f(y)$ (the boundary condition at $x = 1$) to get

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh n\pi \ \ \sin n\pi y.$$

Note that the functions $\phi_n = \sin \pi y$ are orthogonal in $L^2[0, 1]$ (we have shown this several times at this point!). As always, take inner products of both sides with $h_m = \sin m\pi y$ to get the coefficients:

$$\langle f, h_m \rangle = (a_m \sinh m\pi) \langle h_m, h_m \rangle$$

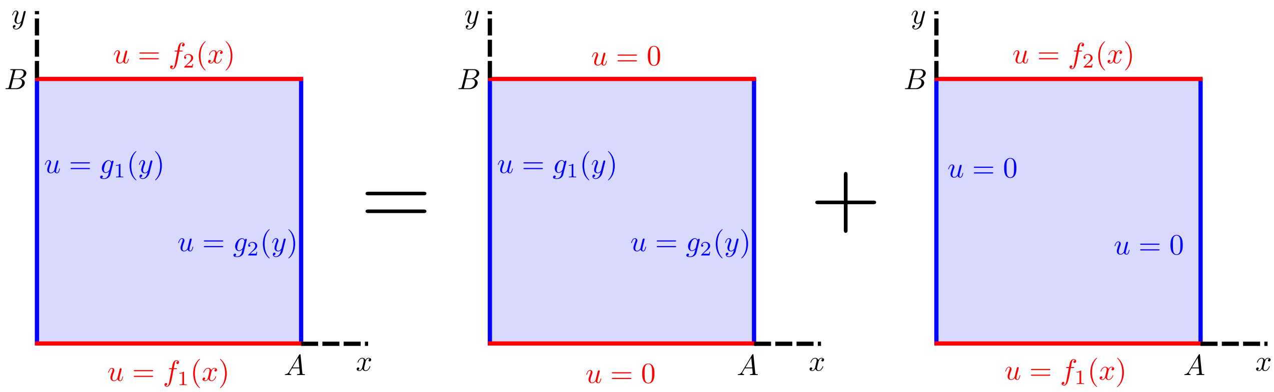
so

$$a_n = \frac{1}{\sinh n\pi} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}, \qquad n \geq 1.$$

2.2 Rectangle, with more boundary conditions

Let's return to the rectangle example and consider how to solve the problem when there are inhomogeneous boundary conditions applied at all the sides for Laplace's equation in a rectangle of width A and height B:

$$\begin{aligned} 0 &= u_{xx} + u_{yy}, \quad x \in (0,a), \quad y \in (0,b) \\ u(x,0) = f_1(x), &\quad u(x,1) = f_2(x), \quad x \in (0,A) \\ u(0,y) = g_1(y), &\quad u(1,y) = g_2(y), \quad y \in (0,B). \end{aligned} \tag{2.2}$$



Both pairs of opposite sides (in blue and red above) could have non-homogeneous BCs. Our method only works if one of those pairs is homogeneous.

To solve (2.2), we use superposition and break the problem up into parts. Each part will take care of one (or two) of the boundaries and leave all the others zero. When added together, the sum of the parts will satisfy all the boundary conditions.

We find v, w solving

$$\begin{aligned} 0 &= v_{xx} + v_{yy}, \quad x \in (0,A), \quad y \in (0,B) \\ v(x,0) = 0, &\quad v(x,B) = 0, \quad x \in (0,A) \\ v(0,y) = g_1(y), &\quad v(A,y) = g_2(y), \quad y \in (0,B). \end{aligned} \tag{2.3}$$

$$\begin{aligned} 0 &= w_{xx} + w_{yy}, \quad x \in (0,A), \quad y \in (0,B) \\ w(x,0) = f_1(x), &\quad w(x,B) = f_2(x), \quad x \in (0,A) \\ w(0,y) = 0, &\quad w(A,y) = 0, \quad y \in (0,B). \end{aligned} \tag{2.4}$$

The sum $u = v + w$ is then the solution to (2.2). The solutions u along with v, w for a specific choice of initial condition are shown in Figure 1.

Solving for v: To solve (3.1), look for a separated solution $v = h(x)\phi(y)$. This leads to the boundary value problem

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi(b) = 0.$$

The solutions are

$$\phi_n(y) = \sin\frac{n\pi y}{B}, \quad \lambda_n = n^2\pi^2/B^2.$$

There are no boundary conditions we can apply for h (both boundaries have inhomogeneous terms), which satisfies

$$h_n'' - \lambda_n h_n = 0,$$

However, we can use suerposition again, splitting

$$v = v_1 + v_2$$

where the two pieces have boundary conditions

$$v_1(0, y) = g_1(y), \quad v_1(A, y) = 0$$

and

$$v_2(0, y) = 0, \quad v_2(A, y) = g_2(y)$$

then adding the two solutions together.

For the first one, we get to apply $g(A) = 0$, which yields

$$h_n = a_n \sinh(\mu_n(A - x)), \qquad \mu_n = n\pi/A$$

and the solution

$$v_1 = \sum_{n=1}^{\infty} a_n \sinh(\mu_n(A - x))\phi_n(y).$$

We can apply the other boundary condition now:

$$v_1(0, y) = g_1(y)$$

$$\implies \sum_{n=1}^{\infty} a_n \sinh(\mu_n A)\phi_n(y) = g_1(y)$$

$$\implies a_n \sinh(\mu_n A) = \frac{2}{B} \int_0^B g_1(y)\phi_n(y)\, dy. \tag{2.5}$$

For the second boundary conditions, we apply $g(0) = 0$, which yields

$$h_n = b_n \sinh(\mu_n x), \quad \mu_n = n\pi/A$$

and then

$$v_2 = \sum_{n=1}^{\infty} b_n \sinh(\mu_n x))\phi_n(y).$$

Now we apply the boundary condition at the other side:

$$v_2(A, y) = g_2(y) \implies \sum_{n=1}^{\infty} b_n \sinh(\mu_n A)\phi_n(y) = g_2(y)$$

$$\implies b_n \sinh(\mu_n A) = \frac{2}{B} \int_0^B g_2(y)\phi_n(y)\, dy. \tag{2.6}$$

The solution for v is then

$$v(x,y) = \sum_{n=1}^{\infty} \Big[a_n \sinh(\mu_n(A-x)) + b_n \sinh(\mu_n x) \Big] \phi_n(y)$$

with the a_n's and b_n's given by (2.5) and (2.6).

Finding w that solves (2.4) is the same process, and one gets a similar expression (left as an exercise). Finally, the solution to the original problem (2.2) is

$$u = v + w$$

which will have the form

$$u = \sum_{n=1}^{\infty} h_n(x)\phi_n(y) + \sum_{n=1]}^{\infty} q_n(y)\psi_n(x)$$

for eigenfunctions $\phi_n(y) = \sin(n\pi y/B)$ and $\psi_n(x) = \sin n\pi x/A$, each used to solve the 'half' problems for v and w separately.

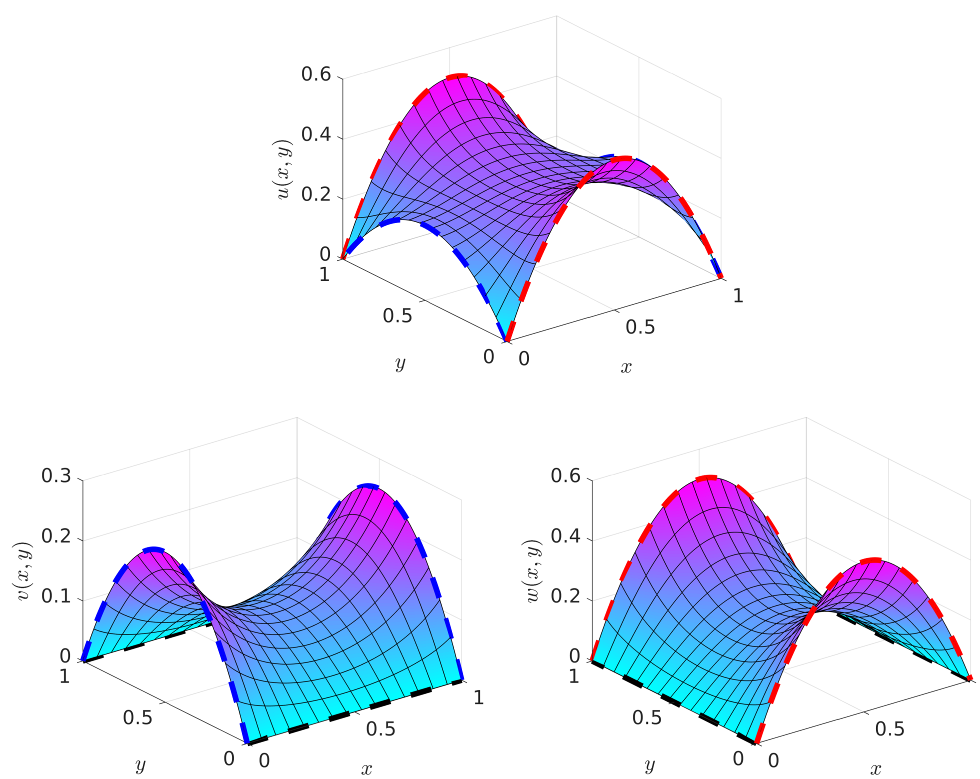


Figure 1: Solution u to (2.2) and v and w to (3.1) and (2.4) for $f_1 = f_2 = x(1-x)$ and $g_1 = g_2 = y(1-y)$.

3 More examples

3.1 A more explicit example

Some solution details, shared by previous details, are omitted - you may fill out the details as you read through. The main point here is to (a) use superposition to make the BCs easier to deal with and (b) use the 'small number of eigenmodes' observation to simplify.

$$\begin{gathered} 0 = u_{xx} + u_{yy}, \quad x \in (0,\pi), \quad y \in (0,\pi) \\ u(x,0) = 1 + \cos 2x, \quad u(x,\pi) = 0, \quad x \in (0,\pi) \\ u_x(0,y) = 0, \quad u_x(\pi,y) = 3\sin y \quad y \in (0,\pi). \end{gathered} \tag{3.1}$$

We use superposition. Let

$$v = \text{solution with only the } x = 0 \text{ BC non-zeroo}$$

$$w = \text{solution with only the } y = \pi \text{ BC non-zeroo}$$

The first half: For v, the BCs in the second line are homogeneous.
We get the eigenvalue problem

$$\phi'(0) = \phi'(\pi) = 0, \quad -\phi''(x) = \lambda\phi(x)$$

for $x \in [0,\pi]$. The eigenfunctions are

$$\phi_n(x) = \cos nx, \quad \lambda_n = n^2, \quad n \geq 0.$$

The solution is then

$$v = \sum_{n=0}^{\infty} c_n(y)\phi_n(x)$$

and we can either use SoV or plug in the series to get

$$c_n''(y) - n^2 c_n(y) = 0.$$

Then, we have one homogeneous BC to apply:

$$v(x,\pi) = 0 \implies c_n(\pi) = 0$$

which yields the solution

$$c_n(y) = a_n \sinh(n(\pi - y)) \text{ for } n \geq 1$$

$$c_0(y) = a_n(\pi - y)$$

Now note that the other BC is

$$v(x,0) = 1 + \cos 2x = \phi_0 + \phi_2$$

so

$$c_0(0) = 1, \; quad c_2(0) = 1, \quad c_n(0) = 0 \text{ otherwise }.$$

The solution then only has two terms:

$$v(x,y) = \frac{\pi - y}{\pi} + \frac{\sinh(2(\pi - y))}{\sinh 2\pi}\cos 2x$$

The other half: Now for w, we have

$$-\psi'' = \lambda\psi, \quad \psi(0) = \psi(\pi) = 0$$

$$h'' - \lambda h = 0$$

for the eigenvalue problem and coefficient ODE. Thus, the eigenfunctions are

$$\psi_n(y) = \sin ny, \quad \lambda_n = n^2, \quad n \geq 0.$$

The solution is

$$w = \sum_{n=1}^{\infty} h_n(x)\psi_n(y)$$

and we get to apply the remaining homogeneous BC:

$$w_x(0,y) = 0 \implies h'n(0) = 0$$

so we can solve the ODE (mostly):

$$h_n(x) = b_n \cosh nx.$$

Now apply the last BC, but notice that

$$w_x(\pi, y) = 3\sin y = 3\psi_1$$

so $h_1(0) = 3$ and all others are zero; thus

$$w(x,y) = 3\cosh nx \sin y$$

The solution: Putting the two pieces together, the solution to the original problem is

$$u = v + w$$

which, written out, is

$$u = \frac{\pi - y}{\pi} + \frac{\sinh(2(\pi - y))}{\sinh 2\pi}\cos 2x + 3\cosh nx \sin y$$