Math 353 Lecture Notes
Intro to PDEs
Eigenfunction expansions for IBVPs

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Topics covered

- Eigenfunction expansions for PDEs
  - The procedure for time-dependent problems
  - Projection, independent evolution of modes

1 The eigenfunction method to solve PDEs

We are now ready to demonstrate how to use the components derived thus far to solve the heat equation. First, two examples to illustrate the proces...

1.1 Example 1: no source; Dirichlet BCs

The simplest case. We solve

$$u_t = u_{xx}, \qquad x \in (0,1),\ t > 0 \tag{1.1a}$$
$$u(0,t) = 0, \quad u(1,t) = 0, \tag{1.1b}$$
$$u(x,0) = f(x). \tag{1.1c}$$

The eigenvalues/eigenfunctions are (as calculated in previous sections)

$$\lambda_n = n^2\pi^2, \quad \phi_n = \sin n\pi x, \qquad n \geq 1. \tag{1.2}$$

Assuming the solution exists, it can be written in the eigenfunction basis as

$$u(x,t) = \sum_{n=0}^{\infty} c_n(t)\phi_n(x).$$

Definition (modes) The n-th term of this series is sometimes called the n-th **mode** or **Fourier mode**. I'll use the word frequently to describe it (rather than, say, 'basis function').

Substitute into the PDE (1.1a) and use the fact that $-\phi_n'' = \lambda_n \phi$ to obtain

$$\sum_{n=1}^{\infty} (c_n'(t) + \lambda_n c_n(t))\phi_n(x) = 0.$$

By the fact the $\{\phi_n\}$ is a basis, it follows that the coefficient for each mode satisfies the ODE

$$c_n'(t) + \lambda_n c_n(t) = 0.$$

Solving the ODE gives us a 'general' solution to the PDE with its BCs,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(x).$$

The remaining coefficients are determined by the IC,

$$u(x,0) = f(x).$$

To match to the solution, we need to also write $f(x)$ in the basis:

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x), \quad f_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = 2\int_0^1 f(x) \sin n\pi x \, dx. \tag{1.3}$$

Then from the initial condition, we get

$$\begin{aligned} u(x,0) &= f(x) \\ \implies \sum_{n=1}^{\infty} c_n(0)\phi_n(x) &= \sum_{n=1}^{\infty} f_n \phi_n(x) \\ \implies c_n(0) &= f_n \text{ for all } n \geq 1. \end{aligned}$$

Now everything has been solved - we are done! The solution to the IBVP (1.1) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \sin n\pi x \quad \text{ with } a_n \text{ given by (1.5).} \tag{1.4}$$

Alternatively, we could state the solution as follows: The solution is

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t} \phi_n(x)$$

with eigenfunctions/values ϕ_n, λ_n given by (1.2) and f_n by (1.3).

1.2 Long-time behavior

Note that every term in the solution (1.4) has a negative exponential (since all the eigenvalues are positive). Furthermore, terms further down in the series decay much faster since λ_n grows quadratically with n. It follows (informally) that

$$\lim_{t\to\infty} u(x,t) = 0$$

independent of the initial condition $f(x)$ (which just affects the b_n's and not the eigenvalues).

Moreover, by approximating u by its first term,

$$u(x,t) \approx f_1 e^{-\lambda_1 t}\phi_1(x) \implies \text{ decays to zero at least as fast as } f_1 e^{-\lambda_1 t}.$$

Thus, the **smallest eigenvalue** of a non-zero term gives the (exponential) convergence rate.

As an explicit example, suppose the initial condition is

$$f(x) = x(1-x).$$

After some laborious integration by parts, we get

$$a_n = \frac{2(1-(-1)^n)}{\pi^3 n^3} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi^3 n^3} & n \text{ odd.} \end{cases} \tag{1.5}$$

The first few terms of the solution are

$$u(x,t) = \frac{2}{\pi^3} e^{-\pi^2 t}\sin \pi x + \frac{2}{27\pi^3} e^{-9\pi^2 t}\sin 3\pi x + \cdots$$

Note that $u(x,t)$ is missing a $\phi_2 = \sin 2\pi x$ term; this does not affect the convergence rate.

Missing modes: However, suppose instead that $f(x) = x - 1/2$. Then we have

$$a_1 = 2\int_0^1 (x-1/2)\sin \pi x\, dx = 0, \quad a_2 = 2\int_0^1 (x-1/2)\sin 2\pi x\, dx = -1/pi.$$

the $n = 1$ term vanishes, so

$$u(x,t) \approx 0\cdot\phi_1(x) - \frac{1}{\pi}e^{-4\pi^2 t}\phi_2(x) + \cdots \text{ as } t\to\infty$$

and the convergence rate is given by the second eigenvalue $\lambda_2 = 4\pi^2$ (faster convergence!).

1.3 Example 2: no source, Neumann BCs

A variation - similar to Dirichlet, but with a crucial difference due to the zero eigenvalue. Here we seek a solution $u(x,t)$ to the IBVP

$$u_t = u_{xx}, \qquad x \in (0,1),\ t > 0 \tag{1.6}$$

with boundary and initial conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \qquad u(x,0) = f(x). \tag{1.7}$$

The eigenvalues/eigenfunctions are (again, computed earlier)

$$\lambda_n = n^2\pi^2, \quad \phi_n = \cos n\pi x, \quad n = 0, 1, 2, \cdots$$

Note that $\lambda_n = 0$ is an eigenvalue, unlike the previous case. Regardless, the process is the same and we end up with a solution (check this!)

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x$$

for constants a_n determined by the initial condition $f(x)$. In terms of the basis,

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x), \qquad \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

However, we must be careful; the formulas are different for $n = 0$ and $n \neq 0$:

$$f_0 = \frac{\int_0^1 f(x)\,dx}{\int_0^1 1\,dx} = \int_0^1 f(x)\,dx,$$

$$f_n = \frac{\int_0^1 f(x)\cos n\pi x\,dx}{\int_0^1 \cos^2 n\pi x\,dx} = 2\int_0^1 f(x)\cos n\pi x\,dx, \qquad n \geq 1.$$

This gives the formulas for hte coefficients a_n in the solution since

$$u(x,0) = f(x) \implies \sum_{n=0}^{\infty} a_n \phi_n = \sum_{n=0}^{\infty} f_n \phi_n.$$

1.4 Long-time behavior (Neumann)

The zero eigenvalue changes the $t \to \infty$ limit. We have

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x = a_0 + a_1 e^{-\pi^2 t} \cos \pi x + \cdots$$

or more abstractly,

$$u(x,t) = a_0\phi_0(x) + a_1 e^{-\lambda_1 t}\phi_1(x) + \cdots$$

As $t \to \infty$, every term with a negative exponential will vanish. However, the $n = 0$ term does not!

Thus, the limit as $t \to \infty$ exists (no terms grow!) and leaves only the $n = 0$ term:

$$\lim_{t\to\infty} u(x,t) = a_0\phi_0(x).$$

We know the $\lambda = 0$ eigenfunction is just $\phi_0 = 1$ and we have a formula for a_0, which gives

$$\lim_{t\to\infty} u(x,t) = \text{constant function}$$

where the value of this constant is

$$a_0 = \int_0^1 f(x)\,dx = \text{ average of } f(x) \text{ over the interval }.$$

The result here makes sense physically. The Neumann problem models heat flow in a closed (insulated) container. Over time, the temperature will reach a (constant) equilibrium, and that value is the average temperature.

2 The projection approach

We have seen that solutions to the heat equation are actually a superposition of **single mode** solutions

$$c_n(t)\phi_n(x)$$

where $c_n(t)$ is governed by a (scalar) ODE. This perspective can be used to guide the solution procedure. Let's go through the procedure for solving the heat equation, but using the idea of projecting onto each mode. Recall that

$$f \to \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \text{ coefficient of the } \phi_n \text{ component of } f.$$

That is, this 'projection' selects the coefficient of the n-th mode.

Consider again the heat equation problem

$$\begin{aligned} u_t =& u_{xx}, \qquad x \in (0,1),\ t > 0 \\ u(0,t) = 0,& \quad u(1,t) = 0, \\ u(x,0) =& f(x). \end{aligned}$$

Find the eigenfunctions $\phi_n = \sin n\pi x$ and eigenvalues $\lambda_n = n^2\pi^2$ as before.

We know that the solution can be written in the eigenfunction basis:

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x).$$

In particular, we've defined $c_n(t)$ as the coefficient of the n-th mode of u. From here, we can project, and then **only have to solve one-dimensional problems** (simple ODEs)!

First, project the PDE by taking the 'projection' onto the ϕ_n component:

$$\text{proj}_{\phi_n}(\cdot) \to \frac{\langle \cdot, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

We calculated before, using the series, that

$$\text{proj}_{\phi_n}(u_t) = c_n'(t), \quad \text{proj}_{\phi_n}(u_{xx}) = -\lambda_n c_n$$

Then for the PDE, we get

$$\text{proj}_{\phi_n}(\text{PDE}) \implies c_n'(t) = -\lambda_n c_n$$

and for the IC,

$$\text{proj}_{\phi_n}(\text{IC}) \implies c_n(0) = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \ \phi_n \text{ coefficient of } f.$$

Thus, the coefficient of the n-th mode evolves according to the IVP

$$c_n' + \lambda_n c_n = 0, \quad c_n(0) = \ \phi_n \text{ coefficient of the IC.}$$

After solving for each mode, we take the superposition to get the full solution.

2.1 Single mode solutions

This principle tells us that for the full problem

$$\begin{aligned} u_t =& u_{xx}, \qquad x \in (0,1),\, t > 0 \\ u(0,t) = 0, \quad & u(1,t) = 0, \\ u(x,0) =& f(x). \end{aligned}$$

each mode

$$u_n(x,t) = c_n(t)\phi_n(x)$$

solves a 'projected' IBVP

$$\begin{aligned} (u_n)_t &= (u_n)_{xx}, \\ u_n & \text{ satisfies the BCs} \\ u_n(x,0) &= f_n\phi_n \end{aligned} \tag{IBVP$_n$}$$

i.e. it solves the heat equation where the 'input' (initial condition) is just the n-th mode of the full IC.

By taking the superposition of the solutions to these projected IBVPs, we get the solution to the full one (since the ICs superimpose to give $f(x)$).

This principle means that if there is only one mode in the inputs, then the solution will also have only one mode. For a simple example, consider

$$\begin{aligned} u_t =& u_{xx} + 4\sin 2x, \qquad x \in (0,\pi),\, t > 0 \\ u(0,t) = 0, \quad & u(\pi,t) = 0, \\ u(x,0) =& 2\sin 2x. \end{aligned}$$

The eigenfunctions are $\phi_n = \sin nx$ for $n \geq 1$. Both the source and IC only have a ϕ_2 term. To be explicit, we have

$$u(x,0) = 2\phi_2 = \sum_{n\geq 1} f_n\phi_n, \qquad f_n = \begin{cases} 2 & n = 2 \\ 0 & n \neq 2 \end{cases}.$$

It follows that only the $n = 2$ mode of the solution is non-zero (all the other projected problems have $u_n = 0$ as their solution), so

$$u(x,t) = c_2(t)\phi_2(x).$$

Indeed, plugging this into the PDE we find that

$$c_2'(t) = -\lambda_2 c_2(t) + 4, \quad c_2(0) = 2 \implies c_2 = 1 + e^{-4t}.$$

No other terms need to be solved for ($c_n(t) = 0$ for $n \neq 2$). The solution is simply

$$u(x,t) = (1 + e^{-4t})\sin 2x.$$

Just to be sure - projection onto the other modes gives

$$c_n'(t) + \lambda_n c_n(t) = 0, \quad c_n(0) = 0 \text{ for } n \neq 2$$

whose solution is just $c_n(t) = 0$. Thus, only the ϕ_2 term of the solution is non-zero.

3 Procedure for the eigenfunction method

The procedure for the heat equation will extend nicely to a variety of other problems. For now, consider an initial boundary value problem of the form

$$\begin{aligned} u_t = -Lu + h(x,t), \quad & x \in (a,b),\ t>0 \\ &\text{hom. BCs at } a \text{ and } b \\ & u(x,0) = f(x) \end{aligned} \tag{3.1}$$

We seek a solution in terms of the eigenfunction basis

$$u(x,t) = \sum_n c_n(t)\phi_n(x)$$

by finding **simple ODEs** to solve for the coefficients $c_n(t)$. This form of the solution is called an **eigenfunction expansion** for u (or 'eigenfunction series') and each term $c_n\phi_n(x)$ is a **mode** (or 'Fourier mode' or 'eigenmode').

Part 1: find the eigenfunction basis. The first step is to compute the basis. The eigenfunctions we need are the solutions to the eigenvalue problem

$$L\phi = \lambda\phi, \qquad \phi(x) \text{ satisfies the BCs for } u. \tag{3.2}$$

By the theorem in **??**, there is a sequence of eigenfunctions $\{\phi_n\}$ with eigenvalues $\{\lambda_n\}$ that form an orthogonal basis for $L^2[a,b]$ (i.e. one with all the required properties).

If possible, we compute solutions explicitly via the standard procedure. Note that the BCs imposed on the eigenvalue problem **must be homogeneous**.

Now at each fixed time t, the function $u(x,t)$ is a function of x defined on $[a,b]$. It follows that there are coefficients $c_n(t)$ such that

$$u(x,t) = \sum_n c_n(t)\phi_n(x). \tag{3.3}$$

For each t,$\{c_n(t)\}$ is the set of coefficients for expressing $u(x,t)$ in terms of the basis $\{\phi_n\}$.

Part 2: get ODEs for the coefficients: Our objective now is to reduce the problem to ODEs for coefficients $c_n(t)$. The order of steps can be changed here.

Step 2a (Write known functions in the basis): We express every known function in the problem (PDE and ICs) in the eigenfunction basis using the orthogonality formula. In this case, there are two such functions, the source term and the initial condition:

$$f(x) = \sum_n f_n\phi_n(x), \qquad f_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}, \tag{3.4}$$

$$h(x,t) = \sum_{n=0}^{\infty} h_n(t)\phi_n(x), \qquad h_n(t) = \frac{\langle h(x,t), \phi_n(x)\rangle}{\langle \phi_n, \phi_n\rangle}. \tag{3.5}$$

Here $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$ is the L^2 inner product in x. Note that in (3.5), the t-variable is not part of the integral, which sometimes simplifies things e.g.

$$h(x,t) = tx \implies \langle h, \phi_n\rangle = t\int_a^b x\phi_n(x)\,dx.$$

Step 2b (Plug series in to the PDE): We now 'plug the series in' to the PDE - which expands the PDE in terms of the eigenfunction basis. Taking this one part at a time[1],

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}\left(\sum_n c_n(t)\phi_n(x)\right) \\ &= \sum_n c_n'(t)\phi_n(x), \qquad (t\text{-derivs. only act on coeffs.}) \\ Lu &= \sum_n c_n(t)L\phi_n \\ &= \sum_n \lambda_n c_n(t)\phi_n(x) \quad (\phi_n \text{ is an eigenfunction}) \end{aligned}$$

. Plugging this into the PDE, we get

$$\begin{aligned} u_t &= -Lu + h(x,t) \\ \implies \sum_n c_n'(t)\phi_n(x) &= -\sum_n \lambda_n c_n(t)\phi_n(x) + \sum_n h_n(t)\phi_n(x). \end{aligned}$$

Since $\{\phi_n\}$ is a basis, the coefficients on each side are equal term-by-term. Or, write

$$\sum_n (c_n'(t) + \lambda_n c_n(t) - h_n(t))\phi_n(x) = 0 = \sum_n 0\cdot\phi_n$$

and since zero must have a unique representation in the basis,

$$c_n'(t) + \lambda_n c_n(t) = h_n(t) \text{ for all } n. \tag{3.6}$$

Next, you can solve the ODE for each n. Note that sometimes, this will involve some case work (as we saw with Fourier series).

We now have a solution to the **PDE with the BCs** in the form

$$u(x,t) = \sum_n c_n(t)\phi_n(x)$$

which will have arbitrary coefficients (a 'general solution').

Part 3: initial conditions, clean up You can do this part earlier as well. The initial condition must be applied. Write

$$u(x,0) = f(x)$$

[1] After getting used to the process, you can shortcut the work here and skip some steps. They tend to be the same for most problems, but you should be careful and recognize when the steps must be changed.

in terms of the basis by plugging in the solution for u and the series for f to get

$$\sum_n c_n(0)\phi_n(x) = \sum f_n\phi_n(x)$$

which gives initial conditions for the c_n's:

$$c_n(0) = f_n. \tag{3.7}$$

That determines the coefficients, yielding a unique solution.

Note: Alternately, you could wait to solve the ODE, then solve (3.6) with (3.7) together:

$$c_n'(t) + \lambda_n c_n(t) = 0, \quad c_n(0) = f_n$$

as an initial value problem (sometimes easier, since you don't need to first find the general solution to the ODE).

Finally, you need to clearly state the solution, collecting the results. A reasonable state would be something like the following:

Example of a solution statement We have that

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\phi_n(x)$$

solves the IBVP (3.1) where

$$\lambda_n = (n\pi/L)^2, \quad \phi_n = \sin(n\pi x/L)$$

and f_n is given by

$$f_n = \frac{2}{L}\int_0^L x^3 \sin(n\pi x)\, dx.$$

Note that it is often best to leave expressions in terms of explicit integrals like f_n above. There's no reason to simplify that integral, unless you really need the numbers.

4 Separation of variables

For **homogeneous** problems, we can exploit this independence to obtain solutions quickly. Not that what follows is a **useful computational trick**, and is justified because of the theoretical framework of eigenfunctions.

4.1 A first example

Consider the equation

$$\begin{gathered} u_t = u_{xx}, \quad x \in [0,\pi],\, t > 0 \\ u(0,t) = u(\pi,t) = 0,\, t > 0 \\ u(x,0) = f(x). \end{gathered}$$

We know that the solution will be an infinite sum of terms (modes) of a certain form. Let us guess a **separated** solution

$$u(x,t) = F(t)G(x).$$

Plug into the PDE to get

$$F'(t)G(x) = F(t)G''(x).$$

Now separate variables, putting all the x's on one side:

$$\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)}.$$

The left side is independent of x and the right side is independent of t. Thus, both must equal the same constant (chosen to be $-\lambda$, knowing it will be the eigenvalue):

$$\begin{gathered} \text{ind. of } x = \text{ind. of } t = \text{const.} \\ \implies \frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = -\lambda. \end{gathered}$$

Plugging this into the boundary conditions, we find that

$$F(t)G(0) = F(t)G(\pi) = 0 \text{ for all } t$$

so we should require

$$G(0) = G(\pi) = 0.$$

This gives a pair of ODEs, linked by a shared constant, one of which has BCs:

$$\begin{gathered} F'(t) = -\lambda F(t), \\ -G''(x) = \lambda G(x), \quad G(0) = G(\pi) = 0. \end{gathered}$$

The problem for G is the eigenvalue problem (for the operator $Lu = -u_{xx}$) with solutions

$$G_n = \sin nx, \quad \lambda_n = n^2 \text{ for } n = 1, 2, \cdots.$$

This sets the possible constants. Now for each constant λ_n, solve for F:

$$\lambda_n \implies F_n = b_n e^{-\lambda_n t}.$$

We have now found all separated solutions:

$$u = F(t)G(x) \text{ solves the PDE + BCs } \iff u = u_n(x,t) = b_n e^{-\lambda_n t} \sin nx.$$

the full solution is then a superposition of the separated solutions (the theory for the eigenfunction basis is required to verify this claim is true):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t).$$

Finally, apply the initial conditions to get the constants b_n (same as before).

Comparison to eigenfunction expansion Separation of variables for a homogeneous PDE is the same steps as the eigenfunction expansion previously used, except construcfted **in a different order**.

Both methods seek solutions for each component/mode, e.g. $c_n(t)\phi_n(x)$ for the heat equation. For **SoV**, we solve for both $c_n(t)$ and $\phi_n(x)$ at once, obtaining a full solution to the PDE/BCs for each n. *Then*, they are added together.

For **eigenfunction expansion**, We find the eigenfunctions, and write the full solution $u(x,t)$ as a superposition of components $c_n(t)\phi_n(x)$. The c_n's are still unknown! Then, we project onto each component and solve for the c_n's.

The method works on some other homogeneous PDEs, as long as tehy can be separateed. Consider the PDE/BCs

$$u_t = u_{xx} + u_x + (t+1)u$$
$$u(0,t) = 0, \quad u(1,t) + u_x(1,t) = 0$$

The PDE is homogeneous, so we can proceed with SoV. Plug in $u = F(t)G(x)$ to get

$$F'(t)G(x) = F(t)G''(x) + F(t)G'(x) + (t+1)F(t)G(x).$$

Divide by $G(x)$ and $F(t)$ to get:

$$\frac{F'(t)}{F(t)} = \frac{G'' + G'}{G} + (t+1).$$

$$\implies \frac{F'(t)}{F(t)} - (t+1) = \frac{G'' + G'}{G} = -\lambda.$$

Now plug into the BCs:

$$F(t)G(0) = 0, \quad F(t)(G(1) + G'(1)) = 0$$

$$\implies G(0) = 0, \quad G(1) + G'(1) = 0.$$

The separated problems are then

$$F' = -(\lambda + t + 1)F$$
$$G'' + G' = -\lambda G, \quad G(0) = G(1) + G'(1) = 0.$$

We then solve this to get eigenfunctions G_n and eigenvalues λ_n, then solve

$$F' + (\lambda_n + t + 1)F = 0 \implies F_n(t).$$

Now use superposition to write the full solution,

$$u = \sum_n F_n(t)G_n(x).$$

The G_n's are the eigenfunctions and the problem for $G(x)$ is the eigenvalue problem.

4.2 Limitations of SoV

When does SoV not yield the solution? Suppose the PDE is **inhomogeneous**, e.g.

$$u_t = 2tu_{xx} + e^x, \quad x \in [0, \pi], \, t > 0$$
$$u(0, t) = u(\pi, t) = 0,$$
$$u(x, 0) = f(x).$$

(Quick reminder: a linear equation $Lu = f$ is inhomogeneous if $f \neq 0$, i.e. there is a term that does not involve u. In particular, it's *homogeneous* if and only if $u = 0$ is a solution).

We cannot find solutions

$$u = F(t)G(x)$$

to the PDE - it is not separable due to the te^x term.

To use the eigenfunction method, we first need to identify the eigenvalue problem. SoV is still useful here! The **homogeneous** problem for the PDE/BCs is

$$u_t = 2tu_{xx} \quad x \in [0, \pi], \, t > 0$$
$$u(0, t) = u(\pi, t) = 0,$$

Using separation of variables we arrive at

$$\frac{F'(t)}{2tF(t)} = \frac{G''(x)}{G(x)} = -\lambda \tag{4.1}$$

and boundary conditions $G(0) = G(\pi) = 0$. The equation for G is

$$G'' = -\lambda G, \quad G(0) = G(\pi) = 0,$$

the eigenvalue problem. The operator here ($LG = \lambda G$) is then

$$L = -\frac{d^2}{dx^2}$$

as before (you would lose the minus sign depending on the sign in (4.1)).

Now, we **go back to the full problem** and use eigenfunction expansion. We now know that the eigenfunctions/eigenvalues are

$$\phi_n = \sin nx, \quad \lambda_n = n^2, \quad n = 1, 2, 3, \cdots .$$

so expand the solution as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

and expand the inhomogeneous term as

$$e^x = \sum_{n=1}^{\infty} h_n \phi_n, \qquad h_n = \frac{\langle e^x, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Then plug both into the PDE:

$$u_t = 2tu_{xx} + e^x$$

$$\implies \sum_{n=1}^{\infty} c_n'(t)\phi_n = 2t\sum_{n=1}^{\infty} c_n(t)(-\lambda_n\phi_n) + \sum_{n=1}^{\infty} h_n\phi_n$$

$$\implies \sum_{n=1}^{\infty}(c_n'(t) + 2t\lambda_n c_n(t) - h_n)\phi_n = 0$$

so (since the ϕ_n's are a basis)

$$c_n'(t) + 2t\lambda_n c_n(t) = h_n, \quad n \geq 1.$$

Notice that SoV yields the same homogeneous part for $c_n(t)$; from (4.1) of the previous example we get

$$F'(t) + 2t\lambda F(t) = 0.$$

The inhomogeneous term contributes an h_n to the ODE for the n-th mode.

Important note: SoV used to identify the operator L and the eigenvalue problem for use with the eigenfunction method when this is not clear. This is because the eigenfunctions for the homogeneous problem are also the correct ones for inhomogeneous problems!

However, when solving the full inhomogenous problem, we do need to return the eigenfunction expansion approach.

5 Wave equation

The wave equation, in one dimension, has the form

$$u_{tt} = c^2 u_{xx}$$

for $u(x,t)$, where c is a constant. This is the fundamental equation for describing propagation of (physical) waves e.g. elecromagnetic, seismic, sonic and so on. As with the heat equation, the wave speed may vary in space. For a vibrating string with variable density $\rho(x)$ and tension T (constant), the displacement $u(x,t)$ evolves according to the PDE

$$\rho(x)u_{tt} = Tu_{xx}.$$

(see e.g. the textbook for a derivation).

5.1 Vibrating string

Consider, for example a string that is fixed at ends $x = 0$ and $x = \ell$ with constant tension T and density ρ. Define the 'wave speed'

$$c = \sqrt{T/\rho}.$$

Then the displacement $u(x,t)$ of the string can be described by the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x \in (0,\ell)$$

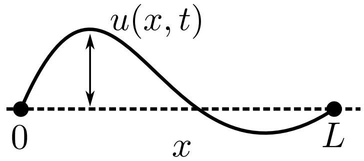
along with boundary conditions

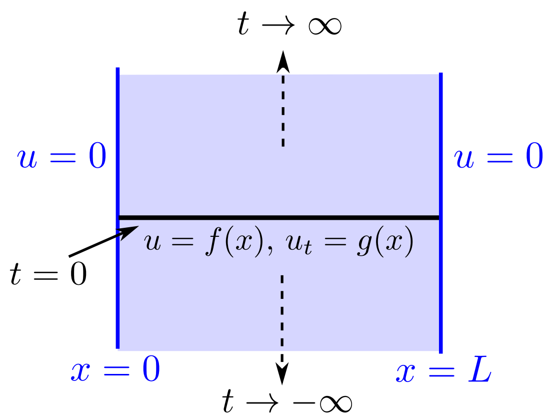
$$u(0,t) = u(\ell,t) = 0.$$

The string has, at $t = 0$, an initial displacement $f(x)$ and velocity $g(x)$. The IBVP is:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x \in (0,\ell),\ t \in \mathbb{R} \\ u(0,t) = 0, &\quad u(\ell,t) = 0, \\ u(x,0) = f(x), &\quad \frac{\partial u}{\partial t}(x,0) = g(x). \end{aligned} \tag{5.1}$$

Note that there are two ICs needed because of the two t-derivatives. A sketch and the domain (in the (x,t) plane) is shown below. We **do not** restrict $t > 0$ as in the heat equation.





5.2 Solution (separation of variables)

Let's solve this homogeneous problem using the full separation of variables procedure.

1) Eigenfunctions, separated solutions: First, look for a separated solution

$$u = h(t)\phi(x).$$

Substitute into the PDE and rearrange terms:

$$u_{tt} = c^2 u_{xx}$$
$$\implies \frac{1}{c^2}\frac{h''(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)}$$

and conclude that both sides must be a constant (neither a function of x nor t):

$$\frac{1}{c^2}\frac{h''(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda. \tag{5.2}$$

Then substitute into the BCs to get

$$\phi(0) = \phi(\ell) = 0. \tag{5.3}$$

Collecting (5.2), (5.3) together, we get an ODE for $h(t)$ and an eigenvalue problem for $\phi(x)$:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(\ell) = 0,$$
$$h''(t) + c^2\lambda h(t) = 0.$$

Solving the ϕ equation as done in the past, we obtain eigenvalues/funcftions

$$\implies \phi_n = \sin\frac{n\pi x}{\ell}, \quad \lambda_n = n^2\pi^2/\ell^2.$$

which identifies the eigenfunction basis for this problem.

Next, we solve the coefficient ODE

$$h_n''(t) + c^2\lambda_n h_n(t) = 0$$

for each λ_n. Plugging in λ_n, we get

$$h_n'' + \omega_n^2 h_n = 0, \qquad \omega_n := \frac{n\pi c}{\ell}.$$

Thus, for all n, the solution consists of sines/cosines, so we can solve for all n at once:

$$h_n = a_n\cos\omega_n t + b_n\sin\omega_n t.$$

In summary, we have found solutions of the form

$$h_n(t)\phi_n(x) = (a_n\cos\omega_n t + b_n\sin\omega_n t)\sin(n\pi x/\ell), \qquad n = 1, 2, 3\cdots.$$

2) Full solution, ICs: The IBVP solution is a superposition of the separated solutions:

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos\omega_n t + b_n \sin\omega_n t)\phi_n(x) \tag{5.4}$$

(note that the a_n's and b_n's from above were arbitrary to begin with, so no new coefficients have to be introduced). This expression is the **general solution to the PDE + BCs**.

Last, the IC,

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x)$$

must be applied to solve the IBVP.

First, write f and g in terms of the eigenfunction basis:

$$f = \sum_{n=1}^{\infty} f_n\phi_n, \quad f_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(n\pi x/\ell)\,dx$$

where $\langle f, g\rangle = \int_0^\ell f(x)g(x)\,dx$ is the L^2 inner product in the interval. Similarly,

$$g = \sum_{n=1}^{\infty} g_n\phi_n, \quad g_n = \frac{\langle g, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{\ell}\int_0^{\ell} g(x)\sin(n\pi x/\ell)\,dx$$

Plugging the series into the ICs gives

$$\sum_{n=1}^{\infty} h_n(0)\phi_n(x) = f(x), \quad \sum_{n=1}^{\infty} h_n'(0)\phi_n(x) = g(x)$$

and projecting onto the ϕ_n component, we get

$$h_n(0) = f_n, \quad h_n'(0) = g_n.$$

That is, the ϕ_n compoment of the solution sees only the ϕ_n compoment of the ICs. It's easy, then, to solve for the unknown coefficients in (5.4) (a_n is just f_n and b_n is g_n).

In summary, the solution to the IBVP is

$$u(x,t) = \sum_{n=1}^{\infty}(f_n \cos\omega_n t + g_n \sin\omega_n t)\phi_n(x)$$

with f_n, g_n as defined above and $\phi_n = \sin(n\pi x/\ell)$ and

$$\omega_n = cn\pi/\ell.$$

5.3 Solution via eigenfunctions

The procedure is similar to SoV. By whatever means, identify the appropriate operator L, which here is

$$L = -d^2/dx^2$$

and we need to solve

$$\begin{aligned} u_{tt} &= -c^2 Lu, \quad x \in (0,\ell),\ t \in \mathbb{R} \\ u(0,t) = 0, &\quad u(\ell,t) = 0, \\ u(x,0) = f(x), &\quad \frac{\partial u}{\partial t}(x,0) = g(x). \end{aligned} \tag{5.5}$$

(the c^2 can be put inside L but it is easier to keep L simple).

The eigenvalue problem (plug in $\phi(x)$ for u) is then

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(\ell).$$

Then we solve for the eigenfunctions/values to get the basis.

Next, everything is expanded in terms of the basis (u and the initial conditions f, g):

$$u = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \quad f = \sum_{n=1}^{\infty} f_n\phi_n, \quad g = \sum_{n=1}^{\infty} g_n\phi_n, \quad f_n = \frac{\langle f,\phi_n\rangle}{\langle \phi_n,\phi_n\rangle},\ g_n = \frac{\langle g,\phi_n\rangle}{\langle \phi_n,\phi_n\rangle}$$

for unknown coefficients.[2] Plug into the PDE to get

$$\sum c_n''(t)\phi_n = -c^2 \sum c_n(t)\lambda_n\phi_n \implies c_n'' + \lambda_n c_n = 0, \quad n \geq 1.$$

and into the ICs to get

$$f(x) = u(x,0) \implies c_n(0) = f_n, \quad g(x) = u_t(x,0) \implies c_n'(0) = g_n.$$

Last, we solve the ODEs for $c_n(t)$ (with the initial conditions above), same as in SoV.

[2] As before, $\langle f, g\rangle = \int_0^\ell f(x)g(x)\,dx$ is the L^2 inner product on $[0,\ell]$.

5.4 Standing waves

The separated solutions (the eigenmodes) in the (PDE+BCs) solution (5.4) have the form

$$u_n(t) = (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin(n\pi x/\ell)$$

The frequencies

$$\omega_1 = c\pi/\ell$$

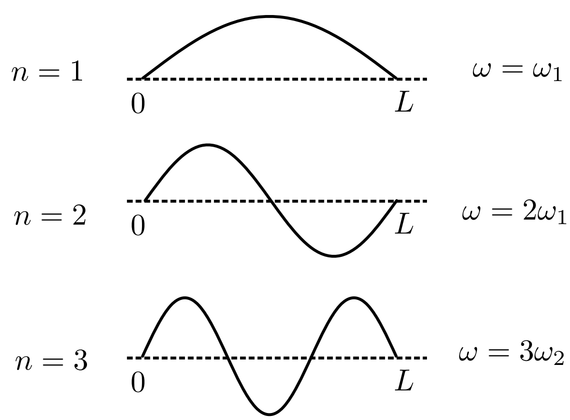
is called the **fundamental frequency**, and the others ($\omega_n = n\omega_1$) are multiples of it.

We have thus shown that, for the wave equation, the motion is a superposition of vibrations at **multiples of the fundamental frequency**.

Notice that if you were to observe a string following the separated solution

$$a_n \cos \omega_n t + b_n \sin \omega_n t)\phi_n(x)$$

you would a see the string vibrate with the shape of the eigenfunction, but with an amplitude that oscillates in time (see below). These are **standing waves**, which you can easily observe in a simple physics experiment.



Where is the wave? So far, it is not clear why the full solution describes a propagating wave. With some effort we can show that the solution to the wave equation is really a superposition of **two** superimposed waves traveling in opposite directions. Using

$$\cos nct \sin nx = \frac{1}{2}(\sin n(x+ct) + \sin n(x-ct)) = \frac{1}{2}h_n(x+ct) + \frac{1}{2}h_n(x-ct)$$

we can rewrite the solution in the form $F(x+ct) + G(x-ct)$ (**D'Alembert's formula**). This hints at key structure for the wave equation (propagation along **characteristics**) that is outside of the scope of the eigenfunction method; we will not pursue it here.

6 Example: plucking a string

A string of length $\ell = 1$ from a guitar or harp is plucked. The initial speed $u_t(x,0) = 0$ and the displacement will a triangular shape like

$$f = 2A \cdot \begin{cases} x & 0 \le x < 1/2 \\ 1-x & 1/2 < x < 1 \end{cases}$$

where A is the initial displacement at $x = 1/2$. Plugging this f into the solution (**??**) (with initial velocity $g = 0$), we find that the response of the string is

$$u(x,t) = \sum_{n=1}^{\infty} f_n \cos(\omega_n t)\phi_n(x), \quad \omega_n = cn\pi, \quad \phi_n = \sin(n\pi x).$$

With some computation, we can find the coefficients $f(x)$,

$$f = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

$$f_n = 2\int_0^1 f(x) \sin n\pi x \, dx = \frac{8A}{\pi^2 n^2} \sin \frac{n\pi}{2} \text{ for } n \ge 1.$$

The even terms are zero! The first few non-zero terms are

$$u(x,t) = \frac{8A}{\pi^2}\left(\phi_1 - \frac{1}{9}\phi_3 + \frac{1}{25}\phi_5 - \cdots\right).$$

The string vibrates with only **odd multiples of the fundamental frequency** (in music terms: the odd harmonics), and the higher frequencies have amplitudes that decay like $1/n^2$.

Further, note that the solution does not look smooth - it is comprised of two waves that move left/right, but retains the sharp corners of the initial pluck more on this later!).

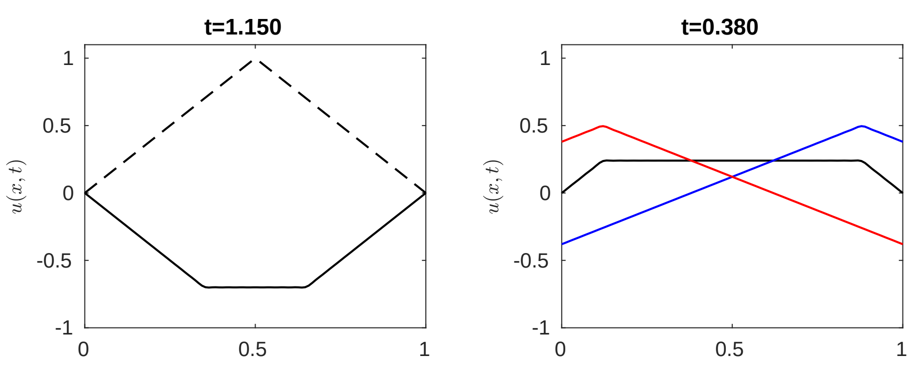


Figure 1: Left: solution and initial condition (dashed). Right: solution and its two waves $h_n(x \pm ct)$ (red and blue) as defind in box on previous page.

6.1 With a source term (tuning the string)

Consider a string of length π, fixed at both ends at take $c = 1$ for simplicity. The system starts at rest, and is then driven by some external force

$$\text{force} = A\sin(Wt)s(x).$$

that oscillates with a frequency W. The IBVP is

$$\begin{aligned} u_{tt} &= u_{xx} + A\sin(Wt)h(x), \quad x \in (0,\pi),\ t \in \mathbb{R} \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 0. \end{aligned} \tag{6.1}$$

Imagine that we are free to control the input frequency W. By 'tuning' this input, we can see the eigenfunctions and eigenvalues by causing it to resonate! (Think like a tuning fork used to tune a musical instrument).

First, let's solve the IBVP using eigenfunctions. Note that there is an inhomogeneous term (a 'source' or 'forcing' term, in other words), so separation of variabels alone cannot be used.

1) Get the eigenfunctions: To identify the eigenfunctions, drop the source term and consider the homogeneous problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad x \in (0,\pi),\ t \in \mathbb{R} \\ u(0,t) &= 0, \quad u(\pi,t) = 0 \end{aligned} \tag{6.2}$$

Procceding with separation of variables ($u = h(t)\phi(x)$) yields the eigenvalue problem

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(\pi) = 0$$

with solutions

$$\implies \phi_n = \sin nx, \quad \lambda_n = n^2, \quad n \geq 1.$$

Thus, the PDE should be viewed as

$$u_{tt} = -Lu, \qquad L = -d^2/dx^2.$$

and the basis used is the basis of eigenfunctions for L with the given BCs.

Now, because the full problem has a forcing term, we must **stop** and return to the eigenfunction method (instead of continuing with SoV).

2) solving the PDE (eigenfunction method): First, write the forcing term in the eigenfunction basis by factoring out the t part and expanding $s(x)$:

$$\text{forcing} = A\sin\omega t \sum_{n=1}^{\infty} s_n\phi_n(x), \quad s_n = \frac{\langle s, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{\pi}\int_0^{\pi} s(x)\phi_n(x)\,dx \tag{6.3}$$

Now write the solution in terms of the eigenfunction basis,

$$u(x,t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x)$$

for unknown functions $h_n(t)$ to be found.

Plug the series for u and the forcing into the PDE to get

$$\sum_{n=1}^{\infty} a_n''(t)\phi_n(x) = \sum_{n=1}^{\infty} h_n(t)\phi_n''(x) + \sum_{n=1}^{\infty} As_n \sin\omega t\,\phi_n(x)$$
$$\implies \sum_{n=1}^{\infty}(h_n''(t) + \lambda_n h_n - As_n \sin\omega t)\phi_n(x) = 0$$

Now for the initial conditions, plug the series for u in (or project onto ϕ_n) to find that

$$\begin{cases} 0 = u(x,0) \implies h_n(0) = 0 \text{ for all } n \\ 0 = u_t(x,0) \implies h_n'(0) = 0 \text{ for all } n \end{cases}.$$

Putting this together, the coefficient $h_n(t)$ of the n-th eigenmode solves

$$h_n''(t) + n^2 h_n(t) = As_n \sin\omega t, \quad h_n(0) = h_n'(0) = 0. \tag{6.4}$$

The solution to the IBVP is

$$u(x,t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x), \qquad h_n(t) = \text{ the solution to (6.4)}. \tag{6.5}$$

With a bit more effort, we can solve the ODE to get an explicit solution. However, due to the forcing term, some care is required (case work!).

6.2 Examples: resonance

Some illustrative cases will suffice to understand how the solution is affected by the source:

With a single mode: Suppose the forcing is

$$\text{forcing}(x,t) = \sin Wt\,\sin 2x.$$

Then the forcing is a multiple of ϕ_2 so the solution (6.5) only has one non-zero eigenmode ($n = 2$):

$$u(x,t) = h_2(t)\phi_2(x).$$

Explicitly, we have that

$$h_n''(t) + n^2 h_n(t) = 0,\ h_n(0) = h_n'(0) = 0 \text{ for } n \neq 2$$

whose solution, of course, is just $h_n(t) = 0$. For the $n = 2$ eigenmode,

$$h_2'' + 4h_2 = \sin Wt, \quad h_2(0) = h_2'(0) = 0$$

There are two cases to consider for undetermined coefficients. If $W \neq 2$, then the usual guess $c_1 \sin Wt + c_2 \cos Wt$ works for the particular solution. After solving the IVP (left as an exercise), we get

$$W \neq 2 \implies h_2(t) = \frac{1}{4 - W^2}(\sin Wt - \frac{W}{2}\sin 2t).$$

However, if $W = 2$ then $\sin Wt$ is a homogeneous solution! instead, the particular solution must be of the form

$$t(c_1 \sin Wt + c_2 \cos Wt)$$

and again, after solving the IVP, we get

$$W = 2 \implies h_2(t) = -\frac{1}{8}\underbrace{t\cos 2t}_{\text{grows!}} + (\text{homogeneous part}).$$

There is **resonance** (linear growth) if $W = 2$, and no resonance otherwise. We can observe this by forcing the string and waiting - if the amplitude keeps growing, resonance is occuring.

With infinit modes: Now suppose the forcing is

$$\text{forcing}(x,t) = \sin \omega t.$$

Then $s(x) = 1$ from the solution, and

$$\text{forcing} = \sin \omega t \cdot \sum_{n=1}^{\infty} s_n \phi_n(x),$$

$$s_n = \frac{2}{\pi}\int_0^\pi \phi_n(x)\,dx = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} 4/n\pi & \text{odd } n \\ 0 & \text{even } n \end{cases}.$$

Then the coefficients $h_n(t)$ for each mode satisfy

$$\begin{aligned} h_n''(t) + n^2 h_n(t) &= s_n \sin Wt, && \text{for odd } n \\ h_n(t) &= 0 && \text{for even } n \end{aligned}$$

Now the undetermined coefficients procedure can be applied to each ODE and we find that

$$h_n = (\cdots) + \text{ homogeneous part if} W \neq n.$$

but

$$h_n = (\cdots)t\cos nt + \text{ homogeneous part if } W = n.$$

Here $\cdots$ denotes a constant you could find, and 'homogeneous' part is a sum of $\sin nt$'s and $\cos nt$'s that just oscillates (does not grow).

Now the full solution has all the (odd) eigenmodes:

$$u(x,t) = \sum_{n=1,\, n \text{ odd}}^{\infty} h_n(t)\phi_n(x).$$

Now let's see what happens if the forcing has frequency W:

- If W is not an odd integer, then there are no resonant eigenmodes. All the $h_n(t)$'s stay bounded, and just oscillate (at frequencies W and n).
- If W is an odd integer ($W = N$), then there is one resonant eigenmode (ϕ_N) and all others stay bounded.

In this second case, because one term grows, it eventually takes over and becomes the dominant term in the solution:

$$u(x,t) = h_N(t)\phi_N(x) + \text{smaller terms as } t \text{ increases.}$$

This means, in particular, that if the string is forced at one of the resonant frequencies,

$$u(x,t) \approx (\text{amplitude}(t) \cdot \phi_N(x) \text{ after some time....}$$

If we force the string with frequency N, then wait a bit, the string will look like the eigenfunction ϕ_N (times some oscillating amplitude).

Thus, we are able to identify the eigenvalues and eigenfunctions by adjusting W to look for each one! An example with $W = 3$ is shown below (note that the amplitude is changing in time; in the plots shown, the snaphots are at times where the amplitude is not large).