Math 353 Lecture Notes
Introduction and some first order ODEs

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Topics covered

- Introduction
- Linearity: linear operators and ODEs
- Solution techniques for some classes of first-order ODEs:

  Separable

  Linear (by integrating factor)
- Ways that a solution can fail to exist; finding the interval of existence

1 Classification of DEs

1.1 Definitions

A differential equation is an equation that describes a function in terms of its derivatives. Such equations are ubiquitous in the sciences, where physical systems depend on the rates of changes of quantities (acceleration, radioactive decay, wave oscillations, population growth rate...).

An **ordinary differential equation** (ODE) relates a function $y(t)$ of one variable to its derivatives. In its most general form, this is

$$F(t, y, y', y'' \cdots, y^{(n)}) = 0.$$

The highest derivative, n, is the **order** of the ODE. We will look only at equations that specify this highest derivative in terms of the others:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}). \tag{1}$$

This is an equation relating *functions*. Without derivatives, e.g.

$$y(t) = t + y(t)^2,$$

the equation can be reduced to a set of scalar equations - for each t solve for $y(t)$. We cannot do this for an ODE (1) because of the derivatives - the value of $y(t)$ is not enough to know $y'(t)$. This coupling makes ODEs challenging to solve (and also gives them with a rich mathematical structure).

A **partial differential equation** (PDE) is an equation for a function of more than one variable involving its *partial* derivatives. A few examples:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ for } u(x,t) \quad \text{(Wave equation)}$$
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ for } \phi(x,y). \quad \text{(Laplace's equation)}$$

The coupling between derivatives in several directions makes PDEs even more challenging to solve than ODEs. We will build up theory for ODEs first, and study PDEs later.

1.2 Linear ODEs

It is difficult to gain much insight into the general ODE (1). Perhaps the most important subclass of ODEs are **linear** ODEs, which have a very specific structure that can be exploited to solve and understand their properties.

1.2.1 Vector spaces

Before defining a linear ODE, we review the notion of a vector space. A vector space is a set of elements ('vectors') V together with scalars (which we will assume to be in $\mathbb{R}$) such that vectors can be added or multiplied by scalars and

$$V \text{ is closed under linear combinations.}$$

That is, if $c_i \in \mathbb{R}$ are scalars and $v_i \in V$ (for $i = 1, \cdots, k$) then

$$\sum_{i=1}^{k} c_i v_i \in V.$$

A **basis** for a vector space is a set $B \subset V$ such that

$$\text{any } v \in V \text{ is a unique linear combination of elements in } B.$$

That is, each v has a unique representation as a linear combination of basis elements.

When the vector space is $\mathbb{R}^n$, its structure is familiar (from linear algebra). However, we are concerned here with vector spaces of *functions*. Note that if f_1 and f_2 are functions from a domain D to $\mathbb{R}$ and c_1, c_2 are scalars then

$$c_1 f_1 + c_2 f_2$$

is also a function (with the same domain). Thus functions are also 'vectors'. Throughout the course, we will try to extend concepts from linear algebra (e.g. bases) to vector spaces of functions.

To properly define a linear ODE, we need the concept of a **differential operator**. This is a special type of function that takes in a *function* as an input, and then outputs another *function* that depends on the input and its derivatives. For now, we will leave the domain/range of L vague (we'll introduce the rigorous notion of *function spaces* later).

The simplest differential operator is

$$L[y] = \frac{dy}{dx}$$

(where $y(x)$ is the input function), which is just the derivative operator. Note that since $L[y]$ is a function this means that

$$L[y] \text{ is the function defined by } L[y](x) = y'(x).$$

An operator is *linear* if

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] \tag{2}$$

for all scalars c_1, c_2 and functions y_1, y_2 in the domain of L. The analogous notion for $\mathbb{R}^n$ is a matrix A, which is a linear operator that takes a vector x and maps it to Ax.

Some example operators:

$$\begin{aligned} L[y] &= yy' && \text{(nonlinear)} \\ L[y] &= t^2 y + \sin t y''' && \text{(linear)} \\ L[y] &= y' + 1 && \text{(nonlinear).} \end{aligned}$$

Note that the operator

$$L[y] = f(x) y^{(k)}$$

is linear (to show: verify (2) directly). We can also take sums of these terms, but not any more without losing linearity. The most general linear differential operator of order n is

$$L[y] = f_0(x) y + f_1(x) y' + \cdots + f_n(x) y^{(n)}.$$

A linear ODE for $y(t)$ is an ODE that can be written as

$$L[y] = g$$

for some function $f(t)$ and linear differential operator L.

A linear ODE is called **homogeneous** if it has the form

$$L[y] = 0$$

and **non-homogenous** if

$$L[y] = f$$

with f a non-zero function. Compare this to the corresponding notion in $\mathbb{R}^n$: the equations $Ax = 0$ and $Ax = b$. Just as you studied the null space of A (solutions to $Ax = 0$) and solutions to $Ax = b$ in linear algebra, we will study the null space

$$\{y : L[y] = 0\}$$

and solutions to $L[y] = f$.
Another version: Here's a slight variation on the above. Consider an operator

$$\begin{aligned} L[y] &= f(y, y', \cdots, y^{(n)}) \\ &= (\text{some function of } y \text{ and derivs. up to order } n. \end{aligned}$$

You can think of L as being a function of the variables

$$y, y', \cdots, y^{(n)}.$$

To be linear, L has to be a linear function of these variables. This means only multiplying by scalars ($y \to ay$) and addition ($ay + by'$) are allowed, and so

$$L \text{ linaer} \iff L[y] = f_0(x)y + f_1(x)y' + \cdots + f_n(x)y^{(n)}$$

i.e. it must be a sum of functions of x times derivatives of y.

For instance,

$$xy + x^2y' = x^3$$

is linear (coefficients x and x^2 for y and y'), while

$$xy + y^2y' = x^2$$

is not (since y^2y' is non-linear in y).

One last check for homogeneous ODEs: Notice that for an ODE

$$L[y] = f,$$

$y = 0$ is a solution if and only if $f = 0$.

Thus, a linear ODE is homogeneous **if and only if** $y = 0$ is a solution.

2 First order ODEs

To start building up the theory, we focus on the first order ODE

$$y' = f(t, y) \tag{3}$$

and the **initial value problem** (IVP)

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{4}$$

To start, we should clearly state what it means to be a solution:

What is a solution? A **solution** to the IVP (4) is a function $y(t)$ such that

i) $y(t)$ is defined in some interval (a, b) containing t_0 and $y(t_0) = y_0$

iii) $y(t)$ satisfies the ODE (3) in (a, b)

The 'solution' to the IVP comes with a domain where it is defined. The largest interval (a, b) where $y(t)$ is defined is called the **interval of existence**.

The **general solution** to the ODE (or 'solution' for short) is the most general expression for $y(t)$ that satisfies the ODE, including arbitrary constants.

Calculus analogy: The simplest first order IVP is one where the RHS depends on just t:

$$y' = f(t), \quad y(t_0) = y_0.$$

The general solution to the ODE can be written as

$$y(t) = C + \int_{t_0}^{t} f(s)\, ds.$$

for any t_0. An initial condition $y(t_0) = y_0$ would determine the constant of integration. The solution to the initial value problem is

$$y(t) = y_0 + \int_{t_0}^{t} f(s)\, ds.$$

When the right hand side depends on y, we cannot simply integrate.

You can think of the solution to an initial value problem as defined 'up to' b and 'down to' a, starting at t_0.

We define the *interval of existence* for a solution $y(t)$ to an IVP (4) to be the largest t-interval (a, b) containing the initial point t_0 such that $y(t)$ is defined. For example,

$$y' = y^2, \quad y(0) = 1$$

has a solution

$$y(t) = \frac{1}{1-t}$$

whose interval of existence is $(-\infty, 1)$. On the other hand, the solution to

$$y' = y, \quad y(0) = 1$$

$y = e^t$, is defined for all $t \in \mathbb{R}$. This is not obvious from just looking at the ODE! We needed to solve the equations to discover that y grows so fast in one case that it 'blows up' as $t \to 1$. In the next section, equipped with a way to solve ODEs exactly, we will see some examples of solutions failing to exist.

Another example: For an example where it fails to exist on both sides, consider

$$y' = 2ty^2, \quad y(0) = a$$

which has solutions of the form

$$y(t) = \frac{1}{C - t^2}.$$

Plugging in $y(0) = a$ we get $C = 1/a$ so

$$y(t) = \frac{1}{1/a - t^2}.$$

If $a < 0$ then $y(t)$ exists for all t.

However, if $a > 0$ then there are asymptotes at $\pm\sqrt{a}$, so the interval of existence is $(-\sqrt{a}, \sqrt{a})$ - whprich can be arbitrarily small!

3 Separable equations

If the first order ODE is simple enough, we can solve it exactly. One technique is called **separating variables**. Let's start with an example. Suppose we want to solve, for $y(x)$,

$$y' = y\cos x. \tag{5}$$

Written out carefully, the ODE reads

$$\frac{dy}{dx} = y(x)\cos x.$$

The idea is to move all the 'y's to one side, and the 'x's to the other. Informally,

$$\frac{1}{y}\,dy = \cos x\,dx$$

$$\int \frac{1}{y}\,dy = \int \cos x\,dx \implies \ln|y| = \sin x + C.$$

The solution then satisfies

$$|y| = Ce^{\sin x}.$$

Since C is arbitrary, the absolute value isn't needed here ($|y|$ is $\pm y$, but C can also be $\pm$), the general solution to (5) is

$$y = Ce^{\sin x}.$$

The steps here are **not rigorous as written** since we can't multiply by dx, but this is easy to fix with some calculus (see below). We should check our answer by 'plugging in':

$$\frac{dy}{dx} = y(x)\cos x$$

$$\text{LHS} = C\frac{d}{dx}(e^{\sin x}), \quad \text{RHS} = Ce^{\sin x}\cos x$$

and by the chain rule, the LHS is $Ce^{\sin x}\frac{d}{dx}(\sin x) = Ce^{\sin x}\cos x$.

Theory: In general, a **separable equation** is an ODE that can be written as

$$f(y)\frac{dy}{dx} = -g(x) \tag{6}$$

for functions f, g (the minus sign is just for later convenience). We can be rigorous about the solution by first letting F, G be anti-derivatives of f and g, i.e.

$$F'(x) = f(x), \quad G'(x) = g(x).$$

Observe that taking the derivative of $F(y(x)$ plus chain rule gives the LHS:

$$\frac{d}{dx}\left(F(y(x))\right) = F'(y)\frac{dy}{dx} = f(y)\frac{dy}{dx}.$$

Let F, G be anti-derivatives of f, g (i.e. $F' = f$ and $G' = g$). Then (6) is

$$(F(y))' = -G'(x).$$

Now integrate, to find that solutions $y(x)$ satisfy

$$F(y) + G(x) = C.$$

Geometric interpretation: Solutions to

$$F'(y)\frac{dy}{dx} = -G'(x)$$

lie on level sets (contours) of

$$\phi(x, y) = F(y) + G(x).$$

Thus, plotting the contours

$$\phi(x, y) = C$$

will show sets of solutions to the ODE. This perspective may help to show where solutions may stop being defined. An example...

Example 1: Consider the IVP

$$y' = -\frac{x}{y}, \qquad y(0) = a$$

for a constant R. This is separable:

$$yy' = -x$$

which we can integrate to obtain the implicit general solution

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C.$$

Solutions therefore follow arcs of circles (level sets of $x^2 + y^2$). Applying the initial condition, the explicit solution is

$$y = \begin{cases} \sqrt{a^2 - x^2} & \text{if } a > 0 \\ -\sqrt{a^2 - x^2} & \text{if } a < 0 \end{cases}.$$

The interval of existence for $y(x)$ is $(-|a|, |a|)$ (note that we do not include $x = \pm R$ because the ODE itself is not well-defined there, even if $y(x)$ is).

We can read from the ODE that solutions may fail to exist, since $|y'| \to \infty$ as $y \to 0$ (assuming $x \neq 0$). The ODE determines a barrier solutions may not cross, and indeed this is clear from a contour plot of $x^2 + y^2$ (draw it!) - if solutions were to continue along the circle, then $y(x)$ would cease to be a function.

The x-interval on which $y(x)$ is defined depends on the initial condition; to find it, we set $y = 0$ in the solution. This gives $x = \pm|a|$, as expected.

Example 2: Consider the pair of IVPs

$$\begin{aligned} y' &= -(1-2x)y\ , \quad y(0) = -1/2 \qquad (A) \\ y' &= \ \ (1-2x)y^2, \quad y(0) = -1/2 \qquad (B). \end{aligned}$$

On what interval are these solutions defined? Unlike Example 1, the ODE function that gives y' is well-defined eveywhere, so there is no problematic value of y. We need to solve the equations to determine where solutions might fail to exist.

For (A):

$$\frac{y'}{y} = -(1-2x)$$

which integrates to

$$\log|y| = -x + x^2 + C.$$

Applying the initial condition and solving for y, we get

$$y = -\frac{1}{2}e^{x^2-x}.$$

(Note that the absolute value was dropped; this is because $y(0) = -1/2$. The solution for $|y|$ suggests that y cannot change sign, so it is safe to have $|y| = -y$ everywhere).

Even though $|y|$ grows very fast, this is defined for all $x \in \mathbb{R}$.

For (B):

$$\frac{y'}{y^2} = 1 - 2x \implies -\frac{1}{y} = x - x^2 + C.$$

This gives the solution

$$y(x) = \frac{1}{(x+1)(x-2)}.$$

The interval of existence is thus $(-1, 2)$. In (A), y' scales with y, which yields something like exponential growth. In (B), y' scales with y^2, causing it to grow *much* faster - so fast that it 'blows up' in a finite interval. We'll see how this can be detected from the ODE when we consider the existence/uniqueness theorem.

Note that the bounds of the interval depend on both t_0 and $y(t_0)$. If instead we required $y(3) = -1/2$, the interval of existence is not $(2, \infty)$. Instead, we have to solve the IVP again:

$$-\frac{1}{y} = x - x^2 + 8.$$

Since $x^2 - x + 8 > 0$ (no real roots), the solution is defined everywhere!

4 First order linear ODEs (Integrating factors)

A first order linear ODE, has the form

$$y' + p(t)y = g(t).$$

This ODE can always be solved exactly by using an **integrating factor**. First, let's consider an example to see how this works:

$$y' + 2y = e^{-t} \tag{7}$$

The trick is to observe that an expression like this can be found by trying a product rule

$$((\text{function of t}) \cdot y)' = \text{thing} \cdot y' + \text{other thing} \cdot y.$$

However, there's a 'missing' factor on the y'. In this case, we have

$$(e^{2t}y)' = e^{2t}y' + 2e^{2t}y = e^{2t}(y' + 2y).$$

Thus we should multiply both sides of (7) by e^{2t}

$$e^{2t}y' + 2e^{2t}y = e^t$$

$$\implies (e^{2t}y)' = e^t.$$

Now we have an equation that can be integrated directly:

$$\int (e^{2t}y)'\,dt = \int e^t\,dt$$

$$\implies e^{2t}y = e^t + C$$
$$\implies y = e^{-t} + Ce^{-2t}.$$

Theory: Now for the general case

$$y' + p(t)y = g(t). \tag{8}$$

Observe that we cannot just integrate (8) because of the py term.
But the LHS looks a bit like the result of a product rule:

$$(\phi y)' = \phi y' + \phi' y.$$

Motivated by this, multiply by a function $\phi(t)$ to be chosen later:

$$\phi y' + p\phi y = \phi g. \tag{9}$$

We seek a function ϕ such that can be 'factored' into the form

$$(\phi y)' = \phi g \tag{10}$$

Expanding this out, we get

$$\phi y' + \phi' y = \phi g$$

so (4) and (10) are the same if

$$\phi' = p\phi.$$

This is separable, so ϕ can be computed:

$$\phi = e^{\int p(t)\,dt}. \tag{11}$$

The function (11) is called the **integrating factor**. Now that we have derived the right integrating factor, we have a process for solving a first-order linear ODE. First, calculate ϕ according to (11). Then:

$$\begin{aligned} y' + py &= g && \text{(original ODE)} \\ \phi y' + \phi p y &= \phi g && \text{(multiply by } \phi\text{)} \\ (\phi y)' &= \phi g && \text{(factor via product rule)} \end{aligned}$$

and then integrate to obtain

$$y = \frac{1}{\phi} \int \phi(t) g(t)\,dt.$$

Note that the product rule step works because we chose the integrating factor to be (11).

For an initial value problem with

$$y(t_0) = a$$

it is usually best to integrate from t_0 to t:

$$y(t) = \frac{1}{\phi(t)} \left(\phi(t_0) y(t_0) + \int_{t_0}^{t} \phi(s) g(s)\,ds \right).$$

You could, of course, solve the general ODE and then find the constant C.

Example 1: Consider the ODE

$$ty' + 2y = 1/t.$$

Divide by t to find that the integrating factor is

$$\phi = e^{\int 2/t\,dt} = e^{2\log t} = t^2.$$

Multiply the original ODE by t to obtain

$$t^2 y' + 2ty = 1.$$

This factors (as it should, by our choice of ϕ) to

$$(t^2 y)' = 1.$$

Integrating yields the general solution $y = \frac{t+C}{t^2}$.

Example 2 : Consider the IVP

$$y' + 2ty = 1, \qquad y(0) = y_0.$$

The integrating factor is $\phi = e^{\int 2t\,dt} = e^{t^2}$. The ODE then factors into

$$(e^{t^2} y)' = e^{t^2},$$

which we integrate from 0 to t to obtain the solution

$$y(t) = e^{-t^2} y_0 + e^{-t^2} \int_0^t e^{s^2}\,ds.$$

Often, the solution contains some nasty integral we can't evaluate explicitly; in that case it is fine to leave it as an integral expression.

Now suppose we are interested the long-term behavior (as $t \to \infty$) of solutions. Observe that $y' < 0$ when $y > 1/2t$ and $y' > 0$ when $y < 1/2t$, suggesting that solutions are pushed towards $y = 1/2t$ as $t \to \infty$. A direction field confirms this (see Figure).

We can conjecture that $y = \frac{1}{2t} +$ (smaller terms) as $t \to \infty$ (informally, we write this as $y \sim 1/2t$). To check this we can estimate the exact solution in the limit $t \to \infty$ (not trivial for the integral expression).