Additional PDE problems
Math 353, Fall 2020

November 15, 2020

Note: The review problems focus on the PDE material, in part because it's the most difficult, and in part because the book has only a few useful problems [see the final guide], whereas its a reasonable resource for *most* of the ODE topics. A few types of PDE problems are also not represented (e.g. Laplace's equation), since they're covered by the book / HW.

R1. Suppose $u(x,t)$ solves the following wave equation problem with a source term, but with both initial conditions equal to zero, like so:

$$\begin{aligned} u_{tt} &= u_{xx} + f(x)\sin 3t, \quad x \in [0,\pi] \\ u(0,t) &= u(\pi,t) = 0 \\ u(x.0) = 0, &\quad u_t(x,0) = 0 \end{aligned}$$

a) Verify, by solving the equation, the statement that the n-th eigenmode of u is zero if and only if the n-th eigenmode of f is zero, i.e.

$$\text{the } \phi_n \text{ term of } u \text{ is zero} \iff \text{the } \phi_n \text{ term of } f(x) \text{ is zero .}$$

b) Under what conditions do all eigenmodes stay bounded for all t?

R2. Solve

$$\begin{aligned} u_t &= 2u_{xx} + g(t), \quad x \in [0,1], t > 0 \\ u(0,t) = 0, &\quad u(1,t) = 0, \quad t > 0 \\ u(x,0) &= \sin 2\pi x. \end{aligned}$$

where $g(t)$ is a continuous function. Do so as follows:

a) Write $u = v + w$ where w solves the IBVP with $g = 0$.

b) Find the solution for w.

c) Find the solution for v. *Hint: it helps to write the source as* $g(t) \cdot 1$.

Hint again: your solution will end up with an integral of g *- use an integrating factor.*

R3. Consider, for a constant A, the IBVP

$$u_t = u_{xx} + Au, \quad x \in [0, \pi],\ t > 0$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = f(x)$$

For which values of A is there a time-independent solution to the PDE + BCs (ignoring the initial condition)?

Answer this question without solving the full PDE for $u(x,t)$. Are you able to identify the solution completely, or is there an unknown left?

R4. (do R3 before (d) and (e)). Consider the PDE and BCs

$$u_t = u_{xx} + Au, \quad x \in [0, \pi],\ t > 0$$
$$u(0,t) = u(\pi,t) = 0$$

where A is a constant along with the initial condition

$$u(x,0) = f(x) = \sum_n f_n \phi_n.$$

You are given the eigenvalue problem $-\phi'' = \lambda\phi$ and eigenvalues/functions

$$\lambda_n = n^2, \quad \phi_n = \sin nx$$

a) Find the solution in the form $u = \sum_n c_n(t)\phi_n(x)$ by plugging the series into the PDE. Give formulas for any coefficients in terms of definite integrals (simplify as much as possible).

b) Suppose $A = 1$. Is there a time-independent solution $w(x)$ to the PDE + BCs? If so, under what conditions on $f(x)$ is it true that

$$\lim_{t\to\infty} u(x,t) = w(x)$$

i.e. that $w(x)$ is a steady state?

c) Answer (b) again for $A = 2$.

d) Answer (b) yet again for $A = 4$.

R5. Suppose $f(x)$ is an L^2 function defined on the interval $[0, \pi]$ and

$$\int_0^\pi f(x)\cos nx\, dx = 0 \text{ for all } n \geq 1.$$

a) What can be said about f?

b) Answer the same question, but if instead

$$\int_0^\pi f(x)\sin nx\, dx = 0 \text{ for all } n \geq 1.$$

R6. Consider the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx}, \quad x \in (0,2),\ t > 0 \\ u_x(0,t) = u_x(2,t) &= 0, \quad t > 0 \\ u(x,0) &= \sin(\pi x/2). \end{aligned}$$

Show that there is a steady state (explain why the limit exists!)

$$w(x) = \lim_{t\to\infty} u(x,t)$$

and compute it (get an explicit result). Note that you will need to solve for $u(x,t)$, but may not need to evaluate all of the constants/coefficients fully.

R7. Consider the problem of finding $u(x,t)$ such that

$$\begin{aligned} u_t + 2\gamma u_x &= u_{xx}, \quad x \in (0,1),\ t > 0 \\ u(0,t) = 0, \quad u(1,t) &= 0, \quad t > 0. \end{aligned}$$

where $\gamma \neq 0$ is a real number.

a) Derive a set of DEs you could solve to find all product solutions $u = c(t)\phi(x)$.

b) State the relevant eigenvalue problem and solve it. Also show that there are no non-oscillatory eigenvalues (i.e. not containing sin or cos).

R8. Consider the IBVP

$$\begin{aligned} u_{tt} &= 4u_{xx} + \sin t \cos x \quad x \in [0,\pi], \\ u_x(0,t) &= 0, \quad u_x(\pi,t) = 0, \\ u(x,0) &= 0, \quad u_t(x,0) = f(x) \end{aligned}$$

a) Derive the ODEs (with initial conditions) you could solve for the coefficients of the solution $u = \sum c_n(t)\phi_n(x)$. *Hint: the problem is not homogeneous!.*

You are given the eigenvalues/functions

$$\cos nx, \quad \lambda_n = n^2, \quad n \geq 0.$$

b) Solve the ODEs to complete the solution if

$$f(x) = 1 + 2\cos x.$$

(Pay attention to which ones are trivial and which you actually need to solve)