HOMEWORK 9
MATH 353, FALL 2020

DUE WEDNESDAY OCT. 28

Book problems:

- Section 10.5: 7

Supplementary problems: 10.5, 7-12, 21 (all heat equation problems)

Non-book problems:

P1. We defined a linear differential equation to be one of the form

$$Lu = f \tag{1}$$

where L is an operator involving derivatives that is linear.

For the heat equation with a source,

$$u_t = ku_{xx} + f \tag{2}$$

what is 'the operator' in the sense of the definition of a linear DE as in (1)? (Note: in class, we used L to denote only part of the operator asked for here).

b) Again, recall that an equation $Lu = f$ is 'homogeneous' if $f = 0$. Which of the following PDEs are homogeneous?

i) $u_t + x^2 u_x = ku_{xx}$, where c, k are constants

ii) $u_t = u_{xx} + Ax$ where A is a constant

iii) $u_{tt} = c^2 u_{xx} + u$, where c is a constant

c) The term 'homogeneous' is defined the same way for boundary conditions - Which of the following boundary conditions are homogeneous?

i) $u(1, t) = -2$

ii) $u(0, t) = -2u_x(1, t)$ (where $u_x = \partial u/\partial x$)

iii) $u(0, t) = u(1, t)$

P2 (calculations for later problems). Express the function

$$f(x) = x, \quad x \in [0, 2]$$

in terms of the eigenfunction bases obtained by solving the eigenvalue problems

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(2) = 0$$

and

$$-\phi'' = \lambda\phi \quad \phi(0) = \phi'(2) = 0.$$

(Note: you don't need to show work for solving the eigenvalue problems, as both showed up in class or homework). You don't need to show work for standard integrals like $\int xe^x\,dx$, $\int x\cos x\,dx$.

P3. Solve the IBVP

$$\begin{aligned} u_t &= 3u_{xx}, \qquad x \in (0,2),\ t > 0 \\ u(0,t) &= 0, \quad u_x(2,t) = 0, \\ u(x,0) &= x. \end{aligned}$$

How much time is required, approximately, for the maximum value of u to reach 10% of its initial value? (*Use the largest term only to estimate this*).

P4. Let $A \neq 0$ be a constant. Consider the IBVP

$$\begin{aligned} u_t &= u_{xx} + Ax, \qquad x \in (0,2),\ t > 0 \\ u(0,t) &= 0, \quad u(2,t) = 0, \\ u(x,0) &= 0. \end{aligned}$$

a) Look for a solution $u = \sum c_n(t)\phi_n(x)$ (using the eigenfunctions you'd use when $A = 0$). Also expand x in the basis and plug into the PDE. Use this to derive ODEs for $c_n(t)$.

Then solve the ODEs to obtain the solution.[1]

b) What is $\lim_{t\to\infty} u(x,t)$? Express in terms of the eigenfunction basis (as $\sum(\cdots)\phi_n$).

c) (optional, to think about) Is there a simpler/easier way to find the limit in (b)? *Hint: the limit is an 'equilibrium' state that does not change in time.*

[1] You can also use the projection approach; i.e. project all the terms in the PDE onto the ϕ_n component.