HOMEWORK 6
MATH 353, FALL 2020

DUE WEDNESDAY OCT. 7

Reading and Book Problems:

- Read the quick review of partial fractions (or review from another source, if you want); its posted to Piazza under Lecture Notes. You may also want to review the summary of Laplace inversion procedures (posted to the same place).
- Section 6.2: 27a
- Section 6.3: 2, 8b

Supplementary problems: Most random problems in sections 6.2 onward are useful for practicing the mechanics (pick a an equation, solve it). Note that answers are easily checked via computer software (e.g. Wolfram Alpha).

Non-book problems:

Suggested exercise (again, nothing to submit). Calculate the inverse Laplace transform of

$$F(s) = \frac{s}{(s^2+1)^2}$$

using complex partial fractions and the trick from the review guide and/or using the convolution theorem as in class.

P1 (comparing the Laplace and direct procedures). *Note: the point here is to review the typical terms you see in Laplace transforms of ODEs and the corresponding solutions in* t*, to internalize the connections.* Consider the ODE

$$y'' + by' + cy = 0.$$

Show that taking the Laplace transform always gives something like

$$Y(s) = \frac{c_1 s + c_2}{p(s)}$$

where $p(s)$ is the **characteristic polynomial**. Knowing this fact gives some intuition for parallels between the direct and Laplace procedures.

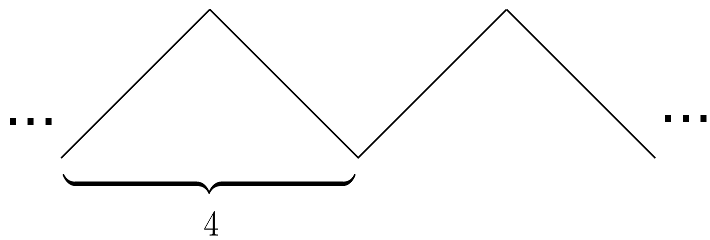
a) Suppose p has two real roots. What terms appear in the partial fraction decomposition for $Y(s)$? Check that this matches the solution from the direct procedure..

b) Now do the same for the repeated roots case, the pure complex roots case ($\pm\omega i$) and complex-with-real-part case ($r \pm \omega i$).

P2. a) Create a triangle wave defined for $t > 0$ with period 4, writing it as a sum of things times step functions:

$$f(t) = \sum_{n=0}^{\infty} (\cdots) u_{\dots}(t)$$

I've sketched one below. Take the height of the triangle to be 2. Note that 'period 4' means it repeats after 4 units of time, i.e. $f(t) = f(t+4)$.



b) Calculate its Laplace transform (you may need to leave as an infinite sum).

P3 (pushing a swing). A swing, modeled as a simple pendulum, is driven to move by a force $g(t)$, starting at rest. Its (angular) displacement is given by

$$y'' + y = g(t), \quad y(0) = y'(0) = 0.$$

a) Calculate the solution **using the Laplace transform** when $g(t) = \cos t$. Use the convolution theorem or (complex) partial fractions to compute the inverse transform.

b) Realistically, you get a swing to move by pushing it once per period of its motion. Such a forcing takes the form of an 'impulse train'

$$g_{tr}(t) = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \cdots = \sum_{n=0}^{\infty} \delta(t - 2\pi n)$$

Find the Laplace transform of the impulse train as an infinite sum.

c) Solve for the response of the swing to this sequence of pushes. *Hint: use the infinite sum and inverse transform term-by-term.*

d) Consider the solution after N pushes have occured. Find an explicit expression for $y(t)$ in this range of time (it should end up being fairly simple) - this gives a nice way of writing the solution.

What is the maximum amplitude of the motion in this range of time? Is there 'resonance' (in the physical sense)?

e) (optional) Calculate the Laplace transform of g_{tr} in **closed form** (use a geometric series). We discussed the connection between poles[1]. of the Laplace transform and resonance. Given the closed form in (b), does the transform suggest resonance?

[1]A *pole* is an asymptote of a *complex*-valued function, e.g. $1/(s - i)$