HOMEWORK 5
MATH 353, FALL 2020

DUE WEDNESDAY, SEP. 23

Book Problems:

- 6.1: 4c, 14 (*For 14, assume things vanish at infinity without proof, and you can use the rule from class instead of the given method if you want.*

Supplementary problems: 3.6: 4, 5, 9, 17 and 6.1: 6, 7, 16, 22.

Non-book problems:

P1. Show that

$$\mathcal{L}[tf(t)] = -F'(s)$$

where $F = \mathcal{L}[f]$ is the Laplace transform of f. Do so by writing down the formula for $F(s)$ and taking the derivative in s. *Hint: you can assume it is allowed to move the derivative inside the integral.*

P2 (a preview; response to forcing). a) Use variation of parameters to find the general solution to

$$y'' - 4y = f(t)$$

where $f(t)$ is a continuous function. (Your solution should be left with integrals). If this is a physical model (e.g. the shape of a bridge pushed by the force of the wind), then you are computing the response of the object to the applied force.

b) The solution can be written as

$$y_p(t) = \text{homogeneous part} + \int_0^t K(t-s)f(s)\,ds.$$

What is K here? As a hint, note that you can do operations like

$$t^2 \int_a^t s\,ds = \int_a^t t^2 s\,ds$$

i.e. you can move functions independent of the integration variable (s) inside the integrand.

c) The variation of parameters formula takes the forcing f as an input, and outputs the 'response' (the particular solution) of the ODE:

$$\text{VOP}[f] = \int_0^t K(s,t)f(s)\,ds.$$

Is this operator linear?

P3 (Another ODE that can be solved exactly). Here, you will adapt the procedure for LCC ODEs to the following ODE:

$$x^2y'' + bxy' + cy = 0, \quad x > 0$$

Here b and c are (real) constants.

a) Follow the idea of the LCC procedure, but use $y(x) = x^\lambda$ instead of $e^{\lambda x}$ and find the general solution to

$$x^2y'' - 4xy' + 6y = 0.$$

b) Do the same for

$$x^2y'' - 3xy' + 8y = 0.$$

Hint: $x^\lambda = e^{\lambda \ln x}$; *also recall Euler's formula.*

c) We saw, for LCC ODEs, two cases that gave solutions e^{rx} and $e^{rx}\sin\omega x, e^{rx}\cos\omega x$. What are the analogous solutions for this type of ODE?

d) Now suppose

$$x^2y'' - xy' + y = 0.$$

Show that your procedure from (a) only yields one solution. Then verify (by plugging into the ODE) that multiplying by $\ln x$ (rather than by x as in $e^{\lambda x} \to xe^{\lambda x}$) gives the other.

Optional: try using reduction of order to obtain the $\ln x$.

P4. Consider the damped oscillator problem

$$x'' + cx' + \nu^2 x = A\cos\omega t$$

a) True or false: when $\omega \neq \nu$, the solution to the ODE is always the sum of oscillations of two different frequencies. (Always here meaning 'for any initial conditions').

b) Now suppose that the system starts at rest:

$$x(0) = x'(0) = 0.$$

Find the solution when $c = 0$ (no damping) and $\omega \neq \nu$. (You don't need to use the rescaling trick here, although you can as long as you convert back to $x(t)$).

c) Do the same as in (b) when $c = 0$ and $\omega = \nu$.

d) Now suppose there is damping - take $c = 1$ and $\nu = 2$ for simplicity. What does the solution look like as $t \to \infty$? Does this answer depend on whether $\omega = \nu$ or not?. (Again, assume the system starts at rest).

Describe the behavior, not just the limit, e.g. $y(t) = e^t + e^{-t}$ looks like e^t as $t \to \infty$.