HOMEWORK 2
MATH 353, FALL 2020

DUE WEDNESDAY, SEP. 2

Notes: Submit the book problems and P1-P2. If you use a theorem, briefly cite it (you can use the theorem number in the book, or a short name like 'the existence theorem'). Make sure that you verify the assumptions of the theorem first.

Important note on supplementary problems: I'll include these (marked as 'supplementary' or with an 'S') as extra problems for practice and study. They are typically mechanical (solution procedures etc.) and related to the homework problems. These problems are never required, and you should not submit them with your homework.

There will also be some non-book supplementary problems (often follow up from lecture, or extra examples that we didn't get to in class, etc.), listed at the end of the homework and also marked with an S.

Book Problems:

- Section 2.4: 5, 7
- Section 2.5: 3

Supplementary problems: 2.4: 6, 8 and 2.5: 1, 6, 7 and S1 (last page)

Non-book problems:

P1. Consider the ODE

$$y' = y(a - y)$$

where a is a **positive** constant. We analyzed this via phase line in class.

a) Solve this ODE by separating variables. Find $y(t)$ explicitly.
Hint: you'll need to use partial fractions; write $1/(y(a-y))$ *as* $c_1/y + c_2/(a-y)$.

b) Now consider the IVP

$$y' = y(a - y), \quad y(0) = y_0$$

when y_0 is **negative**. Determine the interval of existence for the solution and describe the behavior as t increases.

How much of this behavior can you deduce from the phase line, and what requires (a)?

P2. Create an (autonomous) ODE $y' = f(y)$ that has the property that

$$\lim_{t\to\infty} y(t) = 2 \text{ if and } \textbf{only if } 1 < y(0) \leq 2.$$

(Note the 'only if' here!).
You should draw the phase line (and sketch $f(y)$ over it), and create an $f(y)$ that works.

P3 (a substitution trick). (This problem is unrelated to the autonomous equations / existence topics). We saw in class that substituting variables can simplify a problem - here's another example.

a) Show that an ODE for $y(x)$ of the form

$$\frac{dy}{dx} = f(y/x)$$

can be converted to a separable equation by using the 'change of variables'

$$v = y/x.$$

That is, define $v(x) = y(x)/x$ and derive an ODE for $v(x)$.
Hint: plug $x \cdot v(x)$ into the ODE for y, using the product rule on the $\frac{d}{dx}(y(x))$ term.

b) Use this method to find an explicit formula for $y(x)$ that solves

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}, \quad y(2) = 4.$$

Supplementary non-book problems

S1 (*no submission*). Two details e skipped over in class:

a) Consider the initial value problem

$$y' = f(y), \quad y(0) = a$$

and its solution $y(t)$. Check directly that $y(t - t_0)$ solves

$$y' = f(y), \quad y(t_0) = a.$$

When can you conclude that this is *the* solution?

b) We constructed solutions to

$$y' = 2y^{1/2}, \quad y(0) = 0$$

given by

$$y = \begin{cases} 0 & t < a \\ (t-a)^2 & t > a \end{cases}.$$

(i) Check that y' is continuous at $t = a$, so this is a nice, well defined solution to the ODE.

(ii) The separation of variables gave us a solution $(t - a)^2$, but we only kept the right half $(t > a)$. Is this a solution for $t < a$ (check by plugging in)?