HOMEWORK 11
MATH 353, FALL 2020

DUE MONDAY NOV. 16 (TUESDAY IS OK)

Note: The homework score will be calculated using the first problem, plus the maximum of the scores on P1 and P2 - using one to study for the final and only handing in the other for the homework is fine.

Book problems:

- 10.6: 22a (note: you don't have to solve the PDE here, although its a good exercise)

Non-book problems:

P1 (semi-infinite strip). Consider a 'semi-infinite strip'

$$0 < x < \pi/2, \qquad y > 0$$

and Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in that domain, subject to the boundary conditions

$$u(x,0) = f(x), \qquad u_x(0,y) = u_x(\pi/2, y) = 0$$

along with the 'condition at ∞ given by

$$\lim_{y\to\infty} u(x,y) = 0. \qquad x \in (0, \pi/2).$$

a) Draw a sketch of the domain and label the boundary conditions.

b) Derive the form of the solution $u(x,y)$ without applying the $u(x,0) = \cdots$ condition. *Hint: apply the ∞ condition when you solve the coefficient ODE.*

c) Find the solution (and write it out explicitly) when

$$f(x) = 3\cos 2x + \cos 4x.$$

d) Now find the solution for a general $f(x)$.

Are there any restrictions on the $f(x)$'s to have a valid solution?

Note: write $f(x)$ in the form

$$f(x) = \sum_n^\infty f_n \phi_n(x)$$

and give formulas for the f_n's. You will find useful the formula

$$\int_0^{\pi/2} \cos^2(2nx)\, dx = \frac{\pi}{4}, \quad n \geq 1$$

but note that this holds only for $n \geq 1$.

P2 (Laplace in a circle). Consider Laplace's equation **outside** a semi-circle of radius 1:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad & r \in (1,\infty), \quad \theta \in [0,\pi] \\ u(r,0) = u(r,\pi) = 0 \quad & r \in (1,\infty) \\ u(1,\theta) = f(\theta), \quad & \theta \in [0,\pi] \end{aligned}$$

a) Derive the form of the solution (without applyiung the $u(1,\theta)$ condition), using the additional condition that

$$u \text{ is bounded in the domain.}$$

Hint: note that the domain is oustide a circle, so r goes from 1 to ∞. There's an 'inside the semi-circle' example in the lecture notes.

b) Find the solution when

$$f(\theta) = \theta(\pi - \theta).$$

Write out the first two non-zero terms of the solution explicitly. Note that you can, and probably should, use computer algebra to calculate the coefficients (computing the integrals).