Lecture Notes on PDEs, part I: The heat equation and the eigenfunction method

Fall 2018

Contents

Notation: Hereafter, subscripts will denote derivatives with respect to a variable. For instance,

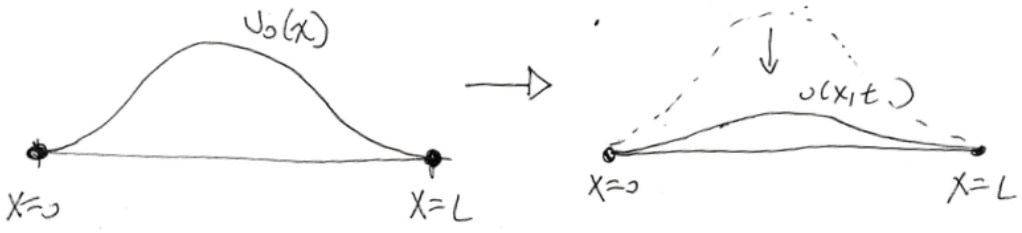
$$u_t(x,t) = \frac{\partial}{\partial t}(u(x,t)), \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad \text{etc.}$$

A **partial differential equation** (PDE) for a function of more than one variable is a an equation involving a function of two or more variables and its partial derivatives.

1 Motivating example: Heat conduction in a metal bar

A metal bar with length $L = \pi$ is initially heated to a temperature of $u_0(x)$. The temperature distribution in the bar is $u(x,t)$. At the ends, it is exposed to air; the temperature outside is constant, so we require that $u = 0$ at the endpoints of the bar.

Over time, we expect the heat to diffuse or be lost to the environment until the temperature of the bar is in equilibrium with the air ($u \to 0$).



Physicist Joseph Fourier, around 1800, studied this problem and in doing so drew attention to a novel technique that has since become one of the cornerstones of applied mathematics. The approach outlined below hints at some of the deep structure we will uncover in the remainder of the course.

The temperature is modeled by the **heat equation** (see subsection 7.1 for a derivation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad t > 0 \text{ and } x \in (0, \pi).$$

Since the temperature is fixed at both ends, we have

$$u(0,t) = 0, \quad u(\pi,t) = 0 \text{ for all } t.$$

Lastly, the initial heat distribution is $t = 0$ is

$$u(x,0) = f(x)$$

where $f(x)$ is some positive function that is zero at 0 and π. The temperature should decrease as heat leaks out of the bar through the ends; eventually it all dissipates. The solution $u(x,t)$ should predict this.

In summary, our goal is to find a function $u(x,t)$ defined on $[0,\pi]$ satisfying

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad t > 0 \text{ and } x \in (0,\pi), \tag{1a}$$

$$u(0,t) = u(\pi,t) = 0 \text{ for } t \geq 0 \tag{1b}$$

$$u(x,0) = u_0(x). \tag{1c}$$

Our objective here is just to find a solution to the first two parts, (1a) and (1b) and worry about the initial condition later.

First, let us guess a solution of the form

$$u = e^{-\lambda t}\phi(x). \tag{2}$$

Substituting into the PDE (1a), we find that

$$-\lambda\phi(x) = \phi''(x).$$

Now substitute into the boundary conditions (1b) (note that $e^{-\lambda t}$ cancels out here) to get

$$\phi(0) = 0, \quad \phi(\pi) = 0.$$

For convenience set $\lambda = \mu^2$. It follows that (2), our guess for u, satisfies the PDE (1a) and the boundary conditions (1b) if the function $g(x)$ solves the **boundary value problem**

$$\phi''(x) + \mu^2\phi(x) = 0, \quad \phi(0) = 0,\ \phi(\pi) = 0. \tag{3}$$

This problem is not an initial value problem (conditions are imposed at both ends), but it is a constant-coefficient ODE, so we can still solve it explicitly. The general solution is

$$\phi = c_1 \sin(\mu x) + c_2 \cos(\mu x).$$

Imposing the condition $\phi(0) = 0$ we find that

$$\phi = c_1 \sin(\mu x).$$

The second condition, $\phi(1) = 0$, requires that

$$\sin(\mu\pi) = 0.$$

Non-trivial solutions exist whenever μ is a non-zero integer. We have now found an infinite sequence of solutions to (3):

$$\phi_n(x) = \sin(nx), \qquad n = 1, 2, 3, \cdots$$

Observe that (3) is a linear, homogeneous problem. In particular,

$$\phi_1, \phi_2 \text{ are solutions to (3)} \implies c_1\phi + c_2\phi_2 \text{ is a solution.} \tag{4}$$

This means that for any constant a_n, the function

$$a_n e^{-n^2 t}\phi_n(x) \tag{5}$$

is a solution to the heat conduction problem with initial data

$$u_0(x) = a_n \sin(nx).$$

Now the crucial question: *what happens when the initial data is not a sine*? No single solution of the form (5) will work. Fourier's breakthrough was the realization that, using the superposition principle (12), the solution could be written as an *infinite* linear combination of all the solutions of the form (5):

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t}\phi_n(x).$$

Then $u(x,t)$ solves the original problem (10) if the coefficients a_n satisfy

$$u_0(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{6}$$

This idea is a generalization of what you know from linear algebra (representing vectors in terms of a basis) but with basis *functions* $\{\sin(nx) : n = 1, 2, 3, \cdots\}$.

In fact, this set of functions has the rather remarkable **orthogonality** property

$$\int_0^\pi \phi_m(x)\phi_n(x)\,dx = \int_0^\pi \sin(mx)\sin(nx)\,dx = 0, \qquad m \neq n. \tag{7}$$

To solve for the coefficient a_m, we can multiply (6) by $\sin(mx)$ and integrate:

$$\int_0^\pi u_0(x)\sin(mx)\,dx = \int_0^\pi \sum_{n=1}^{\infty} a_n \sin(mx)\sin(nx)\,dx.$$

Now move the integral inside the sum (it is not trivial to show this is allowed!). By the property (7), only one of the terms in the sum will be non-zero:

$$\begin{aligned}
\int_0^\pi u_0(x)\sin(mx)\,dx &= \int_0^\pi \sum_{n=1}^{\infty} a_n \sin(mx)\sin(nx)\,dx \\
&= \sum_{n=1}^{\infty} a_n \int_0^\pi \sin(mx)\sin(nx)\,dx \\
&= \left(\sum_{n=1,n\neq m}^{\infty} a_n \cdot 0\right) + a_m \int_0^\pi \sin(mx)\sin(mx)\,dx \\
&= a_m \int_0^\pi \sin^2(mx)\,dx.
\end{aligned}$$

Magically, the infinite sum has been reduced to a simple equation for a_m:

$$a_m = \frac{\int_0^\pi u_0(x)\sin(mx)\,dx}{\int_0^\pi \sin^2(mx)\,dx}. \tag{8}$$

This process works for all m, so the solution to the heat conduction problem (5) with arbitrary initial condition $u_0(x)$ is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t}\sin(nx)$$

with the coefficients given by the formula (8). Of course, all of the manipulations here are formal and unjustified - it is far from clear whether the series converges, or if it is valid to swap integrals and sums, and so on (Fourier did not know this either when first applying the method; it took several decades to settle the issue).

2 The heat equation: preliminaries

Let $[a,b]$ be a bounded interval. Here we consider the PDE

$$u_t = u_{xx}, \quad x \in (a,b),\, t > 0. \tag{9}$$

for $u(x,t)$. This is the **heat equation** in the interval $[a,b]$.

Remark (adding a coefficient): More generally, we could consider

$$u_t = k u_{xx}$$

where $k > 0$ is a 'diffusion coefficient'. However, since the constant can be scaled out by defining a rescaled time $\tau = t/k$ to get

$$u_\tau = u_{xx}$$

there is no loss of generality in studying the structure of (10). Note that it is **essential that the coefficient is positive**; a negative k will produce drastically different results.

2.1 Initial and boundary conditions

An **initial boundary value problem** (IBVP) for the heat equation consists of the PDE itself plus three other conditions specified at $x = a, x = b$ and $t = 0$. As a simple example:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad t > 0 \text{ and } x \in (a,b), \tag{10a}$$

$$u(a,t) = 0 \text{ and } u(b,t) = 0 \qquad \text{for } t > 0 \tag{10b}$$

$$u(x,0) = f(x). \tag{10c}$$

There are three components:

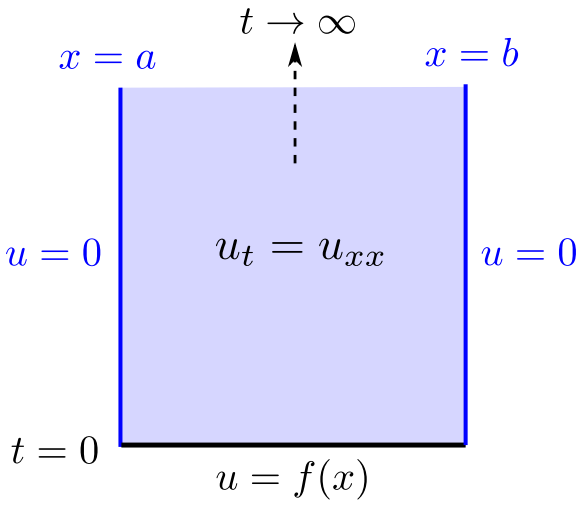
The PDE: Equation (10a) is the PDE (sometimes just 'the equation'), which thThe be solution must satisfy in the entire domain ($x \in (a, b)$ and $t > 0$ here).

Boundary conditions (BCs): Equations (10b) are the **boundary conditions**, imposed at the boundary of the domain (but not the boundary in t at $t = 0$). Each boundary condition is some condition on u evaluated at the boundary.

Initial conditions (ICs): Equation (10c) is the **initial condition**, which specifies the initial values of u (at the initial time $t = 0$).

The initial boundary value problem (10a)-(10c) has a unique solution provided some technical conditions hold on the boundary conditions.

One can think of the 'boundary' of the solution domain to have three sides: $\{x = a\}, \{x = b\}$ and $\{t = 0\}$, with the last side left open (the solution fills this in as $t \to \infty$). The initial condition is really a boundary condition at $t = 0$.[1]



Definition (important BCs): There are three basic types of boundary conditions. Most of the time, we will consider one of these when solving PDEs.

Dirichlet $u(a, t) = 0$ (or 'zero boundary conditions')

Neumann $u_x(a, t) = 0$ (or 'zero flux')

Robin $\alpha u_x(a, t) + \beta u(a, t) = 0$ (or 'radiation')

[1] The three-sided boundary is called the **parabolic boundary** of the IBVP.

The heat equation could have different types of boundary conditions at a and b, e.g.

$$u_t = \alpha u_{xx}, \quad x \in [0,1],\, t > 0$$
$$u(0,t) = 0, \quad u_x(1,t) = 0$$

has a Dirichlet BC at $x = 0$ and Neumann BC at $x = 1$.

Modeling context: For the heat equation $u_t = \alpha u_{xx}$, these have physical meaning. Recall that u is the temperature and $-\alpha u_x$ is the heat flux.

Dirichlet The temperature u is fixed at the end.

Neumann The end is insulated (no heat enters or escapes).

Radiation Some heat enters or escapes, with an amount proportional to the temperature:

$$-\alpha u_x = \beta u.$$

For the interval $[a, b]$ whether heat enters or escapes the system depends on the endpoint and β. The heat flux $-\alpha u_x$ is to the right if it is positive, so at the left boundary a, heat enters the system when $\beta > 0$ and leaves when $\beta < 0$.

Similarly, at the right boundary b, heat enters the system when $\beta < 0$ and leaves when $\beta > 0$.

The same interpretations apply when the equation is describing diffusion of some other quantity (e.g. diffusion of a chemical in a tube).

2.2 Linearity and homogeneous PDEs

The definitions of linear and homogeneous extend to PDEs. We call a PDE for $u(x,t)$ **linear** if it can be written in the form

$$L[u] = f(x,t)$$

where f is some function and L is a linear operator involving the partial derivatives of u. Recall that **linear** means that

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2].$$

The PDE is **homogeneous** if $f = 0$ (so $l[u] = 0$) and **inhomogeneous** if f is non-zero.

Some examples of linear PDEs we will study are

$$u_t = u_{xx} + g(x,t) \qquad (L[u] = u_t - u_{xx}),$$

$$u_{tt} = u_{xx} + g(x,t) \qquad (L[u] = u_{tt} - u_{xx}),$$

$$u_{xx} + u_{yy} = g(x, y) \qquad (L[u] = u_{xx} + u_{yy} = \nabla^2 u),$$

which are the heat equation, wave equation and the Poisson equation, respectively. Note that the function is $u(x, y)$ in the last one. An example of a non-linear PDE would be

$$u_t + uu_x = u_{xx}$$

The same definitions apply to **boundary conditions**. All the boundary conditions listed in the previous section are linear homogeneous. For example,

$$u_x(a, t) = 0 \tag{11}$$

is a linear boundary condition since if u and v satisfy (11) and

$$w = c_1 u + c_2 v$$

then w also satisfies (11) since

$$w_x(a, t) = c_1 u_x(a, t) + c_2 v_x(a, t) = 0.$$

Non-homogeneous boundary conditions can be imposed, for instance

$$u(a, t) = t$$

which might be used to model the ambient temperature increasing with time.

Key fact: A linear, homogeneous PDE obeys the **superposition principle**:

$$u_1, u_2 \text{ are solutions} \implies c_1 u_1 + c_2 u_2 \text{ is a solution} \tag{12}$$

for all scalars $c_1, c_2 \in \mathbb{R}$. The same definition applies to boundary conditions. For instance, all the boundary conditions listed above are linear homogeneous.

Note that an inhomogeneous PDE does not have this property! However, the 'homogeneous' part (i.e. the equation with the inhomogeneous term set to zero) does, and we will find that, as with ODEs, superposition will still be useful.

2.3 More on superposition

The superposition principle (12) is a crucial feature of linear homogeneous problems. Note that while this property is true for homogeneous PDEs and boundary conditions, it is not quite true when initial conditions are included. If u and v are both solutions to the homogeneous problem

$$\begin{aligned} u_t &= u_{xx}, \qquad t > 0 \text{ and } x \in (a, b), \\ u(a, t) &= u(b, t) = 0 \quad \text{for } t > 0 \end{aligned}$$

where u has initial condition

$$u(x,0) = f_1(x)$$

and v has initial condition

$$v(x,0) = f_2(x)$$

then $w = u + v$ solves the IBVP

$$\begin{aligned} w_t = w_{xx}, \qquad & t > 0 \text{ and } x \in (a,b), \\ w(a,t) = w(b,t) = 0 \quad & \text{for } t > 0 \\ w(x,0) = f_1(x) + f_2(x). & \end{aligned}$$

Superimposing two solutions to the PDE with BCs will give another solution, and the initial conditions get superimposed.

We can exploit superposition to split a problem into simpler parts. For example, suppose we seek u solving the inhomogeneous problem

$$\begin{aligned} u_t = u_{xx} + h(x,t), \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = f(x) \end{aligned}$$

with non-zero initial conditions. We can split this into

$$u = v + w$$

where v solves an *inhomogeneous* problem with *zero* initial conditions,

$$\begin{aligned} v_t = v_{xx} + h(x,t), \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = 0 \end{aligned}$$

and $w(x,t)$ solves a *homogeneous* problem with *non-zero* initial conditions,

$$\begin{aligned} w_t = w_{xx}, \\ w(0,t) = w(1,t) = 0, \\ w(x,0) = f(x), \end{aligned}$$

thereby splitting the problem for u into two simpler parts.

To check this, plug $v + w$ into the PDE, boundary conditions and initial conditions and use linearity. For the PDE, we check that

$$u_t = (v+w)_t = v_t + w_t = v_{xx} + w_{xx} + h(x,t) = u_{xx} + h(x,t).$$

For the boundary at $x = 0$, we have

$$u(0,t) = v(0,t) + w(0,t) = 0$$

and similarly $u(1,t) = 0$. Finally, for the initial condition,

$$u(x,0) = v(x,0) + w(x,0) = 0 + f(x) = f(x).$$

3 Orthogonal functions and L^2

Before proceeding to the PDEs, we need to identify the right space in which to consider our solutions and extend notions of orthogonality and so on from linear algebra in $\mathbb{R}^n$ to this space. The main goal here is to extend the notion of an orthogonal basis of eigenvectors in $\mathbb{R}^n$ to a space of functions.

3.1 Linear algebra analogy: using orthogonal bases in $\mathbb{R}^n$

Recall that two vectors $\mathbf{x}$ and $\mathbf{y}$ are called **orthogonal** if their dot product is zero:

$$\mathbf{x} \cdot \mathbf{y} = 0 \qquad \text{where } \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n. \tag{13}$$

An **orthogonal basis** for $\mathbb{R}^n$ is a basis $\mathbf{b}_1, \cdots \mathbf{b}_n$ such that the basis vectors are all pairwise orthogonal, i.e.

$$\mathbf{b}_i \cdot \mathbf{b}_j = 0, \quad i \neq j.$$

An orthogonal basis is convenient because it means that the representation of a vector $x \in \mathbb{R}^n$ in terms of the basis is easy to find.

(Important!) Solving (in $\mathbb{R}^n$) for the coefficients: Suppose $\mathbf{x} \in \mathbb{R}^n$ and $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is an orthogonal basis for $\mathbb{R}^n$. We know that there are unique coefficients $a_1, \cdots a_n$ such that

$$\mathbf{x} = \sum_{j=1}^{n} a_j \mathbf{b}_j.$$

Now consider a single index i. To find the i-th coefficient a_i, we can take the dot product of both sides with $\mathbf{v}_i$. The dot product $\mathbf{v}_i \cdot (\cdots)$ sends all the components except $\mathbf{v}_i$ to zero:

$$\mathbf{b}_i \cdot \mathbf{x} = \sum_{j=1}^{n} a_j (\mathbf{b}_i \cdot \mathbf{b}_j) = a_i \mathbf{b}_i \cdot \mathbf{b}_i$$

so

$$a_i = \frac{\mathbf{b}_i \cdot \mathbf{x}}{\mathbf{b}_i \cdot \mathbf{b}_i}. \tag{14}$$

A key result in linear algebra is the following:

Theorem (spectral theorem; symmetric matrices): Let A be an $n \times n$ real symmetric matrix. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of A are real and distinct, and the corresponding eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ form an orthogonal basis for $\mathbb{R}^n$.

Now suppose we have an $n \times n$ matrix A and wish to solve

$$A\mathbf{x} = \mathbf{b}. \tag{15}$$

Assume A is invertible and real **symmetric**. Using an orthogonal basis of eigenvectors, we can turn the system $Ax = b$ into n *independent* equations that are trivial to solve.

Using the theorem, there is an orthogonal basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ for $\mathbb{R}^n$ of eigenvectors of A, which satisfy

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \text{ for } i = 1, \cdots, n \text{ and } \mathbf{v}_i \cdot \mathbf{v}_j = 0, \text{ for } i \neq j.$$

Write $\mathbf{b}$ in terms of this basis (using the boxed method above):

$$\mathbf{b} = \sum_{i=1}^{n} d_i \mathbf{v}_i, \quad d_i = \frac{b \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}. \tag{16}$$

Now we know that $\mathbf{x}$ (the unknown) also has a representation in the basis,

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i \tag{17}$$

where the c_i's are the coefficients we need to find. Plug (16) and (17) into the system (15) and use that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ to get

$$\sum_{i=1}^{n} \lambda_i c_i \mathbf{v}_i = \sum_{i=1}^{n} d_i \mathbf{v}_i.$$

Since the $\mathbf{v}_i$'s form a basis, it must be that the coefficients of each term on the left and right hand side are equal, so the above implies that

$$c_i = d_i/\lambda_i, \qquad \text{for } i = 1, \cdots n.$$

The solution to (15) is therefore

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i, \qquad c_i = \frac{1}{\lambda_i} \frac{b \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}.$$

3.2 What is the right space of functions?

We need to define a notion of orthogonality for functions. Let us consider **real** functions defined on an interval $[a, b]$. Define the **inner product**

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx \tag{18}$$

can call two functions f, g **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

Analogous to the Euclidean norm for a vector $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$$

we define the L^2 norm

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

Finally, we define the relevant space of functions:

Definition (L^2 spaces)

$$L^2[a,b] = \{f : [a,b] \to \mathbb{R} \text{ such that } \|f\|_2 < \infty\}$$

or equivalently, the set of functions f on $[a,b]$ such that the integral of its square is finite,

$$\int_a^b |f(x)|^2 \, dx < \infty.$$

The space $L^2[a,b]$ is a vector space and the inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

is well-defined for all $f, g \in L^2[a,b]$.

The L^2 norm gives us a way to measure distance between two functions. The expression

$$\|f - g\|_2^2 = \int_a^b |f(x) - g(x)|^2 \, dx \tag{19}$$

is a sort of weighted measure of the area between the curves $f(x)$ and $g(x)$ on the inteval $[a,b]$. This is analogous to the Euclidean distance for vectors:

$$\|\mathbf{x} - \mathbf{y}\|^2 = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$$

which is the actual distance in $\mathbb{R}^n$ between the points at $\mathbf{x}$ and $\mathbf{y}$.

The quantity (19) is sometimes called the **mean-square distance** or **mean-square error** if g is some approximation to f.

Warning (complex functions): All the definitions here are true only for **real-valued functions**. For complex-valued functions, the inner product is instead

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \, dx$$

where $\overline{g(x)}$ is thet complex conjugate of $g(x)$. Most of the theory is the same, other than the occasional conjugate.

3.3 Some examples; importance of the interval

For functions f and g, notions like orthogonality **depend on the underlying space** where the functions live. For instance, consider

$$f(x) = 1, \quad g(x) = \cos x.$$

Regarded as functions in $L^2[0, \pi]$, the two functions are orthogonal:

$$\langle f, g \rangle = \int_0^\pi \cos x \, dx = \sin x \Big|_0^\pi = 0.$$

However, as functions in $L^2[0, \pi/2]$ the two functions are not orthogonal, since then

$$\langle f, g \rangle = \int_0^{\pi/2} \cos x \, dx = 1.$$

The functions are orthogonal on $[0, \pi]$ but not on $[0, \pi/2]$; the domain matters because the definition of the inner product is different for each.

Another example: Consider the space $L^2[-1, 1]$. We have that

$$\langle 1, x \rangle = \int_{-1}^1 x \, dx = 0$$

so the constant function 1 and x are orthogonal on $[-1, 1]$. However,

$$\langle 1, x^2 \rangle = \int_{-1}^1 x^2 \, dx = \frac{2}{3},$$

so 1 and x^2 are not orthogonal. On the other hand, for $g(x) = x^2 - 1/3$,

$$\langle 1, g \rangle = \int_{-1}^1 (x^2 - 1/3) \, dx = \frac{2}{3} - \frac{2}{3} = 0.$$

This means that the set

$$\{1, x, x^2 - 1/3\}$$

is an orthogonal set in $L^2[-1, 1]$, whereas $\{1, x, x^2\}$ is not. The process, incidentally, can be continued to generate an orthogonal sequence of polynomials.

4 Eigenfunctions

Let us return to the heat equation in a bounded domain with Dirichlet boundary conditions:

$$u_t = u_{xx}, \quad x \in (a,b), \quad t > 0 \tag{20}$$
$$u(a,t) = u(b,t) = 0, \quad t > 0. \tag{21}$$

Now write the PDE in the form

$$u_t = -L[u], \tag{22}$$

where L is the linear operator

$$L[u] = -u_{xx}$$

or, in operator notation,

$$L = -\frac{\partial^2}{\partial x^2}.$$

Now let us regard L as an operator acting on functions $\phi(x)$ (just functions of x), i.e.

$$L[\phi] = \phi'' \text{ or } L = -\frac{d^2}{dx^2}.$$

Definition (eigenfunction): We say that ϕ is an **eigenfunction** of the problem (20) with boundary conditions (21) if it solves the **eigenvalue problem**

$$L[\phi] = \lambda\phi, \quad \phi(a) = 0, \, \phi(b) = 0 \tag{23}$$

for some $\lambda \in \mathbb{R}$ (the 'eigenvalue'). Equivalently, we say that ϕ is an eigenfunction of the operator L on $[a,b]$ with boundary conditions $\phi(a) = 0$, $\phi(b) = 0$.

Note: the boundary conditions (21) can be replaced with some other conditions; the definition is the same. The boundary conditions for ϕ are the result of plugging $u = \phi(x)$ into the boundary conditions for u.

Important note: To have an eigenfunction of the operator L, we must prescribe an interval $[a,b]$ and associated boundary conditions.

For instance, for the problem

$$u_t = u_{xx}, \quad x \in (0,\pi) \quad t > 0,$$
$$u(0,t) = u_x(\pi,t) = 0, \quad t > 0,$$

the eigenvalue problem is

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi'(\pi) = 0$$

and we say ϕ is an eigenfunction for $L = -d^2/dx^2$ on $[0,\pi]$ with Neumann boundary conditions (or, explicitly, 'with boundary conditions $\phi'(0) = \phi'(\pi) = 0$').

Notice that the eigenvalue problem is an **ODE**, so we are really studying a type of ODE problem and making use of it (later) to solve PDEs. In the following section, we forget about the PDE part for now and find some eigenfunctions in typical cases.

Notation (why the negative sign?): The negative sign is just convention and is not necessary; we could instead write

$$u_t = L[u], \qquad L = \frac{\partial^2}{\partial x^2}.$$

The ODE for the eigenvalue problem is then

$$\phi'' = \lambda\phi$$

which is the same as the eigenvalue problem for $L = -\partial^2/\partial x^2$, with the sign of λ reversed since

$$-\phi'' = (-\lambda)\phi.$$

Since λ **is an unknown anyway**, it does not matter; we will just get λ's that differ by a negative sign and the eigenfunctions will be the same.

The reason for using the negative sign is that it tends to make most, if not all, the eigenvalues positive (rather than mostly/all negative); see examples below.

4.1 Solving the eigenvalue problem

Consider the operator

$$L = -\frac{d^2}{dx^2}.$$

The eigenvalue problem for L in any interval with any boundary conditions is straightforward to solve, since we can find the general solution exactly. First, let us consider the problem

$$L\phi = \lambda\phi, \quad \phi(0) = \phi(1) = 0 \tag{24}$$

i.e. Dirichlet boundary conditions in the interval $[0, 1]$. The [problem is

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(1) = 0. \tag{25}$$

The characteristic polynomial for the ODE is

$$p(r) = r^2 + \lambda.$$

The roots are

$$r = \pm\sqrt{-\lambda}.$$

There are three cases to consider, as the form of the solution depends on the roots of p.

Case 1 ($\lambda < 0$): If λ is negative, then the two roots $\pm\sqrt{-\lambda}$ are both real and distinct. it follows that the general solution for the ODE is

$$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}.$$

For convenience, set $\mu = \sqrt{-\lambda}$ (not necessary, but saves space), so that

$$\phi = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

We want values of $\mu > 0$ such that ϕ solves (25). Now impose the boundary conditions. First,

$$0 = \phi(0) = c_1 + c_2 \implies c_1 = -c_2,$$

so

$$\phi = c_1(e^{\mu x} - e^{-\mu x}).$$

Now impose the boundary condition at $x = 1$:

$$0 = \phi(1) = c_1(e^{\mu} - e^{-\mu}).$$

If $c_1 = 0$ then $c_2 = 0$ which makes the solution trivial. So we only have a (non-trivial) solution if the term in parentheses is zero, which simplifies to

$$e^{2\mu} = 1.$$

But $e^{(\cdots)} = 1$ only if the exponent is 0, so this requires $\mu = 0$, which is impossible because $\mu = \sqrt{-\lambda}$ and λ was assume to be negative.

Case 2 ($\lambda = 0$): In this case the ODE is

$$\phi'' = 0$$

whose solution is

$$\phi = c_1 x + c_2.$$

Imposing the boundary conditions, we find that

$$\phi(0) = 0 \implies c_2 = 0, \quad \phi(1) = 0 \implies c_1 + c_2 = 0$$

so there are no non-trivial solutions.

Case 3 ($\lambda > 0$): Now there are solutions! The roots of the characteristic polynomial are

$$r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}.$$

Let $\mu = \sqrt{\lambda}$. Then the general solution to the ODE is

$$\phi = c_1 \sin(\mu x) + c_2 \cos \mu x.$$

Imposing $\phi(0) = 0$, we find that

$$0 = \phi(0) = \mu c_2 \implies c_2 = 0$$

so $\phi = c_1 \sin(\mu x)$. Imposing the other boundary condition, we find that ϕ is a solution to (25) with $\lambda > 0$ if and only if

$$\sin \mu = 0.$$

This has solutions for $\mu = n\pi$ $(n = 1, 2, \cdots)$, so $\lambda = \mu^2 = n^2\pi^2$. We therefore have eigenfunctions/eigenvalues

$$\phi_n = \sin(n\pi x), \quad \lambda_n = n^2\pi^2, \quad \text{for } n = 1, 2, \cdots .$$

Notes on the mechanics: Some points to note when computing eigenfunctions:

- If ϕ is an eigenfunction so is any scalar multiple since eigenvlaue problems are always linear homogeneous. We always end up multiplying by an arbitrary constant later, so it does not matter which multiple you choose (e.g. $\sin(n\pi x)$ or $2\sin(n\pi x)$.
- There are sometimes reasons to choose a particular scale for each ϕ_n so that it has some nice property (we'll use this for Fourier series later).
- Setting $\lambda = \mu^2$ is just to avoid writing $\sqrt{-\lambda}$ or $\sqrt{\lambda}$.
- The cases are usually but **not always** λ positive/zero/negative.

4.2 Eigenfunctions and orthogonal bases

The process of finding eigenfunctions for the operator $L\phi = \phi''$ is similar for any of the linear homogeneous BCs - the three standard types we defined earlier. All of them (at an endpoint a) have the form

$$\alpha\phi(a) + \beta\phi'(a) = 0 \tag{26}$$

e,g. Dirichlet means $\beta = 0$ and Neumann means $\alpha = 0$.

It turns out, rather miraculously, that the eigenfunctions we produce are - under suitable technical conditions - an orthogonal basis for a certain space. One version of the theorem[2] is as follows (we'll generalize a bit later).

Informally, it says that nice eigenvalue problems will give an orthogonal basis for functions that satisfy the boundary conditions.

[2]Note: the theorem is a consequence of a **spectral theorem** for L^2-like spaces, a rather deep result in analysis/linear algebra that generalizes of the spectral theorem for symmetric matrices of subsection 3.1. As it is well beyond the scope of the course, you will have to take this result on faith.

Sturm-Liouville Theory, simple version: Consider the operator

$$L = -\frac{d^2}{dx^2}$$

in an interval $[a, b]$ and the eigenvalue problem

$$L\phi = \lambda\phi, \qquad \text{with BCs of the form (26) at } x = a \text{ and } x = b. \tag{27}$$

Then the set of eigenfunctions $\{\phi_n\}$ solving (27) forms an orthogonal basis for the space

$$\{f \in L^2[a, b] : \ f \text{ satisfies the BCs } \}.$$

Moreover, it is true that

i) There is one eigenfunction for each eigenvalue,

ii) There are infinitely many positive eigenvalues that increase to ∞,

iii) There are finitely many negative eigenvalues.

We call the set of eigenvalues the **spectrum** of the operator L (in the interval $[a, b]$ with the associated BCs).

The last three points just confirm what we observed in the examples (we will see an example with negative eigenvalues later!). The key result is that the eigenfunctions form an orthogonal basis, which is what we will use in practice to solve PDEs (for now, the other points are just reassurance, as we will typically solve the eigenvalue problem explicitly anyway).

4.3 Example: Neumann boundary conditions

Now we solve the problem with Neumann boundary conditions:

$$L\phi = \lambda\phi, \quad \phi'(0) = \phi'(1) = 0 \tag{28}$$

where $L = -d^2/dx^2$ as before. The process is the same, and in this case, the general solutions are also the same (see previous example).

Case 1 ($\lambda < 0$): The general solution is

$$\phi(x) = c_1 e^{\mu x} + c_2 e^{\mu x}$$

where $\mu = \sqrt{-\lambda}$. We want values of $\mu > 0$ such that ϕ solves (28). First,

$$0 = \phi'(0) = \mu c_1 - \mu c_2 \implies c_1 = c_2,$$

so

$$\phi = c_1(e^{\mu x} + e^{-\mu x}).$$

Now drop the c_1 (since it multiplies the whole thing, it does not matter) and impose the boundary condition at $x = 1$:

$$0 = \phi'(1) = e^{\mu} + e^{-\mu}.$$

Since both e^{μ} and $e^{-\mu}$ are positive, there are no solutions.

Case 2 ($\lambda = 0$): **Different from the previous example!** We have

$$\phi'' = 0 \implies \phi = c_1 x + c_2.$$

Imposing the boundary conditions, we find that

$$\phi'(0) = 0 \implies c_1 = 0, \quad \phi'(1) = 0 \implies c_1 = 0$$

so there is a non-trivial solution

$$\phi = \text{const.}.$$

A flat line of any value has zero derivative at the boundaries. Let $\lambda_0 = 0$ be the eignvalue; the eigenfunction is then

$$\phi_0 = 1$$

(or any other constant we like).

Case 3 ($\lambda > 0$): Set $\mu = \sqrt{\lambda}$. The general solution to the ODE is

$$\phi = c_1 \sin(\mu x) + c_2 \cos \mu x.$$

Imposing $\phi'(0) = 0$, we find that

$$0 = \phi'(0) = \mu c_1 \implies c_1 = 0$$

so

$$\phi = c_2 \cos \mu x. \tag{29}$$

Imposing the other boundary condition, we get

$$0 = \phi'(1) = -c_2 \mu \sin \mu$$

Since $\mu \neq 0$ by assumption, this has solutions for $\mu = n\pi$ ($n = 1, 2, \cdots$). This gives eigenvalues $\lambda = \mu^2$ and eigenfunctions (29):

$$\phi_n = \cos(n\pi x), \quad \lambda_n = n^2\pi^2, \quad \text{for } n = 1, 2, \cdots .$$

The full set of eigenvalues and eigenvectors (collecting all the cases) is

$$\lambda_0 = 0, \quad \phi_0 = 1, \quad \text{and} \quad \lambda_n = n^2\pi^2, \, \phi_n = \cos(n\pi x) \quad \text{for } n = 1, 2, \cdots .$$

Further notes on mechanics (indexing conventions): For some problems, there are positive eigenvalues, a zero eigenvalue, and possibly negative eigenvalues. In the one above, we had a zero eigenvalue and set

$$\lambda_0 = 0, \quad \phi_0 = 1.$$

Typically, the zero index is saved for the zero eigenvalue (if it exists).

With negative eigenvalues, one scheme is to label with negative indices. For instance, if there are eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots \text{ and two negative eigenvalues}$$

then one could label them as

$$\lambda_{-2} < \lambda_{-1} < 0$$

with eigenfunctions ϕ_{-2} and ϕ_{-1}. Or, one could write them as

$$\alpha_1, \alpha_2 \text{ with eigenfunctions } \psi_1, \psi_2$$

or just different labels α and β. Of course, one could also just relabel indices and label all of them $\lambda_1, \lambda_2, \cdots$ but this tends to mean we lose nice formulas like $\sin(n\pi x)$.

What matters is that we can get the full set of eigenvalues and eigenfunctions; labeling is just a nuisance.

5 The eigenfunction method to solve PDEs

5.1 The method (for the heat equation)

We are now ready to demonstrate how to use the components derived thus far to solve the heat equation (and by extension, related PDEs). Consider an initial boundary value problem of the form

$$u_t = -L[u] + h(x,t), \quad x \in (a,b),\ t > 0 \tag{30}$$

with **homogeneous** BCs at a and b (one of the standard ones) and initial condition

$$u(x,0) = f(x).$$

First step: find the eigenfunctions. The eigenfunctions we need are the solutions to the eigenvalue problem

$$L[\phi] = \lambda\phi, \qquad \phi \text{ satisfies the BCs for } u. \tag{31}$$

By an appeal to the theorem in subsection 4.2, there is a sequence of eigenfunctions $\{\phi_n\}$ with eigenvalues $\{\lambda_n\}$ that form an orthogonal basis for the space

$$V = \{f \in L^2[a,b] : f \text{ satisfies the BCs}\}.$$

Now at each fixed time t, the function $u(x,t)$, regarded as a function of x, lies inside the space V. It follows that there are coefficients $a_n(t)$ such that

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x). \tag{32}$$

For each t,$\{a_n(t)\}$ is the set of coefficients for expressing $u(x,t)$ in terms of the basis $\{\phi_n\}$.

Second step: get everything in terms of the basis, substitute into the PDE: Our objective now is to determine the functions $a_n(t)$. First, we write the source h in terms of the eigenfunctions:

$$h(x,t) = \sum_{n=0}^{\infty} h_n(t)\phi_n(x).$$

Now substitute this and the eigenfunction expansion (32) for u into the PDE (30) to obtain

$$\underbrace{\sum_{n=0}^{\infty} a_n'(t)\phi_n}_{u_t} = \underbrace{-\sum_{n=0}^{\infty} a_n(t)\lambda_n\phi_n(x)}_{-L[u]} + \underbrace{\sum_{n=0}^{\infty} h_n(t)\phi_n(x)}_{h}. \tag{33}$$

In detail, the second term was found using the eigenfunction property and linearity of L:

$$\begin{aligned} L[u] &= L\left[\sum_{n=0}^{\infty} a_n(t)\phi_n(x)\right] \\ &= \sum_{n=0}^{\infty} a_n(t)L[\phi_n(x)] \\ &= -\sum_{n=0}^{\infty} a_n(t)\lambda_n\phi_n(x). \end{aligned}$$

Note that since $a_n(t)$ is only a function of t, it is constant as far as L is concerned so by linearity, $L[a_n(t)\phi_n(x)] = a_n(t)L[\phi_n(x)]$. where we have used the fact that L is linear to move

Third step: rearrange to get a single sum: Now collect all the terms in (33) together:

$$0 = \sum_{n=0}^{\infty} (a_n'(t) + \lambda_n a_n(t) - h_n(t))\phi_n(x).$$

Since the ϕ_n's are a basis, the coefficient of each basis function must be zero at all times t (otherwise, the ϕ_n's would be linearly dependent at some t). It follows that for each n,

$$a_n'(t) + \lambda_n a_n(t) = h_n(t) \text{ for } t > 0. \tag{34}$$

This equation is a first-order linear ODE for $a_n(t)$ that is easy to solve. If we want the 'general' solution to the PDE with BCs, then we solve for the $a_n(t)$'s and are done. To solve

the IBVP, the last missing piece is the initial condition - the value of $a_n(0)$.

Find the coefficients at the initial time: To find $a_n(0)$, write the initial condition $f(x)$ in terms of the eigenfunction basis:

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x).$$

For the solution $u(x,t)$ (32) to satisfy the initial condition, we need for constants γ_n. Then it follows that

$$\underbrace{\sum_{n=0}^{\infty} a_n(0)\phi_n(x)}_{u(x,0)} = \underbrace{\sum_{n=0}^{\infty} f_n \phi_n(x)}_{f(x)}.$$

Again, since the ϕ_n's are absis, the two sums must be equal term-by-term, so

$$a_n(0) = f_n.$$

Finally, this condition and (34) lets us solve for a unique $a_n(t)$ (as we get a first order IVP), which completes the process.

5.2 Dirichlet boundary conditions

The simplest case. As an example, suppose we want to find the solution $u(x,t)$ to

$$u_t = u_{xx}, \qquad x \in (0,1),\ t > 0 \tag{35}$$

with boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad u(x,0) = f(x). \tag{36}$$

The eigenvalues/eigenfunctions are (as calculated in subsection 4.1)

$$\lambda_n = n^2\pi^2, \quad \phi_n = \sin n\pi x, \qquad n \geq 1.$$

Assuming the solution exists, it can be written in the eigenfunction basis as

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x).$$

for unknown coefficients $c_n(t)$ (which vary with time). Substitute into the PDE (35) and use the fact that $-\phi_n'' = \lambda_n \phi$ to obtain

$$\sum_{n=0}^{\infty} (a_n'(t) + \lambda_n a_n(t))\phi_n(x) = 0.$$

Equating coefficients of each basis function (to zero), we find that

$$a_n'(t) + \lambda_n a_n(t) = 0. \tag{37}$$

The solution is

$$a_n(t) = a_n(0)e^{-\lambda_n t}.$$

To get $a_n(0)$, write $f(x)$ also in terms of the eigenfunctions:

$$f(x) = \sum_{n=0}^{\infty} b_n \sin n\pi x.$$

Then $u(x,0) = f(x)$ holds if $a_n(0) = b_n$.

To compute b_n, take the inner product of each side with $\sin m\pi x$ to get

$$b_n = \frac{\int_0^1 f(x)\sin n\pi x\,dx}{\int_0^1 \sin^2 n\pi x\,dx} = 2\int_0^1 f(x)\sin n\pi x\,dx. \tag{38}$$

The solution to the problem (35),(36) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin n\pi x \qquad \text{with } b_n \text{ given by (40).} \tag{39}$$

Important special case: If, for example,

$$u(x,0) = f(x) = \sin \pi x + 2\sin 2\pi x$$

then its expansion in the eigenfunction basis,

$$\sin \pi x + 2\sin 2\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

is trivial to find as it is already in the form of a linear combination of basis functions; the coefficient b_1 of $\sin \pi x$ is 1, the coefficient of $\sin 2\pi x$ is 2 and all the others are zero.

It follows that only the $n = 1$ and $n = 2$ terms of the solution series (39) are non-zero; the solution is

$$u(x,t) = e^{-\pi^2 t}\sin \pi x + 2e^{-4\pi^2 t}\sin 2\pi x.$$

Note that this means that given the IBVP with (**??**) as initial conditions, after finding the eigenfunctions/values we could have just assumed the solution had the form

$$u(x,t) = a_1(t)\sin \pi x + a_2(t)\sin 2\pi x$$

and then substituted into the ODE, because we know only these two terms will be non-zero in the end (see homework for further details).

Long-time behavior: Note that every term in the solution (39) has a negative exponential (since all the eigenvalues are positive). Furthermore, terms further down in the series decay much faster since λ_n grows quadratically with n. It follows (omitting issues with the infinite series) that

$$\lim_{t\to\infty} u(x,t) = 0$$

independent of the initial condition $f(x)$ (which just affects the b_n's and not the eigenvalues.

More strongly, we can approximate u by its first term and conclude that

$$u(x,t) \text{ decays to zero at least as fast as } Ce^{-\lambda_1 t}.$$

The **smallest eigenvalue** determines how slow the solution can decay to zero.

If the eigenfunction expansion of $f(x)$ has infinitely many terms, we are stuck with an infinite series solution. For example, suppose

$$f(x) = x(1-x).$$

After some laborious integration by parts, we get

$$b_n = \frac{2(1-(-1)^n)}{\pi^3 n^3} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi^3 n^3} & n \text{ odd.} \end{cases} \tag{40}$$

The first few terms of the solution are

$$u(x,t) = \frac{2}{\pi^3} e^{-\pi^2 t} \sin \pi x + \frac{2}{27\pi^3} e^{-4\pi^2 t} \sin 2\pi x + \cdots$$

5.3 Neumann boundary conditions

A variation - similar to Dirichlet, but with a crucial difference due to the zero eigenvalue. Here we seek a solution $u(x,t)$ to the IBVP

$$u_t = u_{xx}, \qquad x \in (0,1),\ t > 0 \tag{41}$$

with boundary and initial conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \qquad u(x,0) = f(x). \tag{42}$$

The eigenvalues/eigenfunctions are (as calculated in subsection 4.3)

$$\lambda_n = n^2\pi^2, \quad \phi_n = \cos n\pi x, \quad n = 0, 1, 2, \cdots$$

Note that $\lambda_n = 0$ is an eigenvalue, unlike the previous case. Regardless, the process is the same and we end up with a solution (check this!)

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x$$

for constants a_n determined by the initial condition:

$$\sum_{n=0}^{\infty} a_n \cos n\pi x = f(x).$$

Using the orthogonality of the eigenfunctions on $[0,1]$, we get

$$a_n = \frac{\int_0^1 f(x)\cos n\pi x\,dx}{\int_0^1 \cos^2 n\pi x\,dx} = 2\int_0^1 f(x)\cos n\pi x\,dx, \qquad n \geq 1.$$

To find a_0, note that the eigenfunction $\phi_0 = 1$ is orthogonal to the others. Thus, multiplying by $\phi_0 = 1$ and integrating from 0 to 1 we get

$$a_0 = \int_0^1 f(x)\,dx.$$

Convergence as $t \to \infty$: The zero eigenvalue is significant; note that the solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos n\pi x = a_0 + \text{ exp. decaying terms.}$$

As $t \to \infty$, the solution will approach the constant a_0 (again at least as fast as the first term in the series, i.e. $\sim e^{-\lambda_1 t}$). The constant a_0 is the average value of the initial distribution $f(x)$.

This result confirms the intuitive notion that if you put something that diffuses into a closed container (e.g. tea in water), then over time the concentration of stuff will even out until it is uniform.

5.4 An example

To give a more concrete example: suppose we have a mug of water and place a teabag in it, then let it sit. We want to know how long it will take for the tea to diffuse through so that it is uniformly mixed. Assume the mug is one dimensional with height 2 and let $c(x,t)$ be the concentration of tea at height x and time t.

The concentration obeys the heat ('diffusion' in this case) equation

$$c_t = kc_{xx}, \quad x \in [0,1],\ t > 0 \tag{43}$$

where $k > 0$ is the diffusivity of the tea. Since tea cannot leave the cup, the flux at the top and bottom must be zero, so impose Neumann boundary conditions

$$c_x(0,t) = c_x(1,t) = 0, \quad t > 0.$$

Suppose the teabag starts at the bottom of the cup ($x = 0$), and the initial concentration is

$$c(x,0) = f(x) := \begin{cases} 1 & x < 1/2 \\ 0 & x > 1/2 \end{cases}.$$

The eigenvalues/eigenfunctions: It is easiest to write

$$c_t = -kL[c], \quad L = -\frac{\partial^2}{\partial x^2}$$

rather than putting k in the definition of L (but that works too). The eigenvalue problem is

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi'(1) = 0.$$

This is solved as in subsection 5.3 to obtain

$$\lambda_0 = 0, \quad \phi_0 = 1$$

and[3]

$$\lambda_n = (\pi n)^2, \quad \phi_n = \cos(\pi n x), \quad n = 1, 2, \cdots.$$

Eigenfunction expansion and coefficients for the solution: Now that we have a basis, we can write the solution $c(x,t)$ in the form

$$c(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x).$$

Plug this into the PDE (43) and use the property that $-\phi_n'' = \lambda_n\phi_n$ to get

$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) = k\sum_{n=0}^{\infty} a_n(t)\phi_n'' = -k\sum_{n=0}^{\infty} a_n\lambda_n\phi_n$$

and then gather into a single sum:

$$\sum_{n=0}^{\infty}(a_n'(t) + k\lambda_n a_n(t))\phi_n(x) = 0.$$

We conclude that

$$a_n'(t) + k\lambda_n a_n(t) = 0, \quad n \geq 0.$$

For $n = 0$ the eigenvalue is just $\lambda_0 = 0$ so

$$a_0' = 0 \implies a_0(t) = b_0$$

for a constant b_0. For the others, use an integrating factor $e^{k\lambda_n t}$ to get

$$a_n(t) = b_n e^{-k\lambda_n t}$$

for arbitrary constants b_n. The solution to the PDE with the boundary conditions is then

$$c(x,t) = b_0 + \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t}\phi_n(x).$$

[3] If you had taken $L = k\partial^2/\partial x^2$ instead, then the result would be $\lambda_n = k(\pi n)^2$ with the same eigenfunctions since the eigenvalue problem is then $-\phi'' = (\lambda/k)\phi$.

Initial condition: The coefficients are chosen to satisfy the initial condition:

$$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \phi_n(x).$$

The eigenfunctions are orthogonal on $[0, 2]$; the inner product is

$$\langle f, g \rangle = \int_0^2 f(x) g(x)\, dx.$$

Note that

$$\langle \phi_n, \phi_n \rangle = \int_0^2 \cos^2(n\pi x)\, dx = 1 \text{ for } n \geq 1$$

so

$$b_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \int_0^1 f(x) \cos(n\pi x)\, dx = \int_0^{1/2} \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x)\Big|_0^{1/2} = \frac{1}{n\pi} \sin(n\pi/2).$$

for $n \geq 1$ and (taking the inner product with $\phi_0 = 1$)

$$b_0 = \frac{\int_0^1 f(x)\, dx}{\int_0^1 1\, dx} = \int_0^1 f(x)\, dx = 1/2.$$

To summarize, the solution is

$$c(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t} \phi_n(x)$$

where

$$\lambda_n = (n\pi)^2, \quad \phi_n = \cos(n\pi x)$$

and the coefficients are given as above. The first few coefficients are $b_1 = 1/\pi, b_2 = 0$ and $b_3 = -1/3\pi$, so

$$c(x, t) = \frac{1}{2} + \frac{1}{\pi} e^{-k\pi^2 t} \cos(\pi x/2) - \frac{1}{3\pi} e^{-k(3\pi)^2} \cos(3\pi x) + \cdots.$$

Analysis: Since $\lambda_n > 0$ for $n \geq 1$, all the modes decay except the zero mode (the b_0 term) so it follows that

$$\lim_{t \to \infty} c(x, t) = b_0 = \frac{1}{2}.$$

The value the the limit is the **average value** of $f(x)$ in the domain. The amount of tea is constant, so as it diffuses, it converges to a uniform concentration equal to the average in the cup.

The rate of convergence is determined by the first term; we have

$$\max_{x \in [0,1]} |c(x, t) - 1/2| \sim \frac{1}{\pi} e^{-k\pi^2 t}$$

so the (exponential) rate of convergence is $k\pi^2$.

5.5 Another example: finite number of modes

Here, we use three methods of increasing simplicity to solve

$$\begin{aligned} u_t &= 2tu_{xx} + e^{-t^2}\sin x, \quad x \in (0,\pi), \quad t > 0 \\ u(0,t) &= u(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= 3\sin 2x \end{aligned} \tag{44}$$

by exploiting superposition.

Eigenfunctions: The $2t$ goes outside the operator L:

$$u_t = -2tL[u] + h(x,t), \quad h = e^{-t}\sin x$$

where

$$L = -\frac{\partial^2}{\partial x^2}$$

as before. Note that we cannot put t in the definition of L, since L must only involve derivatives in x. The eigenvalues/functions are

$$\lambda_n = n^2, \quad \phi_n = \sin(nx), \quad n = 1, 2, \cdots$$

as found before.

Excessive method:

We could use the general process described in subsection 5.1, starting with

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

and writing

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x),$$

then substituting into the PDE and so on. In this case, the full series is not necessary!

More efficient approach (but still excessive):

Observe that $h(x,t)$ only has an $n = 1$ mode (just a $\sin x$ term). Explicitly,

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x) \text{ where } h_1(t) = e^{-t} \text{ and } h_n(t) = 0 \text{ otherwise .}$$

The initial condition only has an $n = 2$ mode. It follows from the 'general' process in subsection 5.1 than **the solution only has these two modes**. Thus, we know the solution to the IBVP has the form

$$u(x,t) = a_1(t)\sin x + a_2(t)\sin 2x.$$

The rest of the steps are the same as the general case, but with only two terms instead of the whole sum.

Equations for the coefficients: Plug into the PDE:

$$a_1'(t)\sin x + a_2'(t)\sin 2x = -2ta_1(t)\sin x - 2ta_2(t)\sin 2x + e^{-t^2}\sin x.$$

This gives

$$(a_1'(t) + 2ta_1(t) - e^{-t^2})\sin x + (a_2'(t) + 2ta_2(t))\sin 2x = 0.$$

Thus

$$a_1' + 2ta_1 = e^{-t^2} \implies a_1 = b_1 e^{-t^2} + te^{-t^2}$$

using the integrating factor e^{t^2}. For the second mode,

$$a_2' + 2a_2 = 0 \implies a_2 = b_2 e^{-2t}.$$

Initial condition: Plugging $u(x,t)$ into the initial condition, we get

$$3\sin 2x = u(x,0) = a_1(0)\sin x + a_2(0)\sin 2x = b_1 \sin x + b_2 \sin 2x$$

so

$$b_1 = 0, \quad b_2 = 3.$$

The solution is therefore

$$u(x,t) = te^{-t^2}\sin x + 3e^{-t^2}\sin 2x.$$

Only the $n = 1$ mode is affected by the inhomogeneous term; only the $n = 2$ term is affected by the initial condition.

5.5.1 The efficient approach:

Note that **each mode evolves independently**, so we could solve for each term on its own and then add them together using superposition. The solution has the form

$$u = u_1 + u_2$$

where u_1 and u_2 are the $n = 1$ and $n = 2$ modes.

We know that u_1 is the solution to the full problem (46) with all the terms dropped except the $n = 1$ mode:

$$\begin{aligned} u_t &= 2tu_{xx} + e^{-t^2}\sin x, \quad x \in (0,\pi), \quad t > 0 \\ u(0,t) &= u(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= 0 \end{aligned} \tag{45}$$

The solution to this IBVP has only one term, so we can assume that

$$u_1(x,t) = a_1(t)\phi_1(x).$$

Now this form is quite easy to plug into the PDE; we get

$$a_1'(t)\phi_1 = -2ta_1\lambda_1\phi_1 + e^{-t^2}\phi_1$$

which gives

$$a_1'(t) + 2ta_1 - e^{-t^2} = 0.$$

Plugging into the initial condition, we get $a_1(0) = 0$; solving then gives $a_1(t)$ and

$$u_1(x,t) = te^{-t^2}\sin x.$$

Similarly, the full problem (**??**) with only the $n = 2$ mode:

$$\begin{aligned} u_t &= 2tu_{xx} + 0, \quad x \in (0,\pi), \quad t > 0 \\ u(0,t) = u(\pi,t) &= 0, \quad t > 0 \\ u(x,0) &= 3\sin 2x \end{aligned} \tag{46}$$

which has a solution

$$u_2(x,t) = a_2(t)\sin 2x.$$

Again, plug in to get $a_2' + 2ta_2 = 0$ and $a_2(0) = 3$ and solve to get

$$u_2(x,t) = 3e^{-t^2}\sin 2x.$$

The full solution to the original IBVP is then the sum of solutions for each mode,

$$u = u_1 + u_2 =$$

Exploiting independence of modes: The process here illustrates that we can *solve for each mode independently* in the following way:

- Find the eigenvalues and eigenfunctions.
- Expand the inhomogeneous term and initial condition in the eigenfunction basis.
- Solve the problem with only one mode to get a solution $u_n(x,t)$.
- Sum them up to get the full solution.

The full solution is the **superposition of the solutions for each mode.**. When we use the eigenfunction method in subsection 5.1, we are assembling the series from the start,

$$u = \sum a_n(t)\phi_n,$$

then substituting into the PDE to get equations for all of the modes at once. However, they do not interact with each other, so equations like

$$u = \sum (a_n'(t) + \lambda_n a_n(t))\phi_n = 0$$

are really the same as saying that for each n,

$$u_n(x,t) = a_n(t)\phi_n(x) \text{ is a solution if } (a_n' + \lambda_n(t)a_n)\phi_n = 0.$$

It is often easier computationally to compute them one at a time and sum at the end.

6 Separation of variables

For **homogeneous** problems, we can exploit this independence to obtain solutions quickly. Not that what follows is a **useful computational trick**, and is justified because of the theoretical framework of eigenfunctions already developed.

6.1 A first example

Consider the equation

$$u_t = u_{xx}, \quad x \in [0, \pi],\, t > 0$$
$$u(0,t) = u(\pi,t) = 0,\, t > 0$$
$$u(x,0) = f(x).$$

We know that the solution will be an infinite sum of terms (modes) of a certain form. Let us guess a **separated** solution

$$u(x,t) = F(t)G(x).$$

Plug into the PDE to get

$$F'(t)G(x) = F(t)G''(x).$$

Now separate variables, putting all the x's on one side:

$$\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)}.$$

But the left hand side is a function of t and the right hand side is a function of x, so for them to be equal, they must both equal a constant:

$$\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = -\lambda.$$

This gives the ODEs

$$F'(t) = -\lambda F(t), \quad G''(x) = -\lambda G(x).$$

Plugging this into the boundary conditions, we find that

$$F(t)G(0) = F(t)G(\pi) = 0 \text{ for all } t$$

so we should require

$$G(0) = G(\pi) = 0.$$

The problem for G is then

$$G'' + \lambda G = 0, \quad G(0) = G(\pi) = 0$$

which is exactly the eigenvalue problem. It has solutions

$$G_n = \sin nx, \quad \lambda_n = n^2 \text{ for } n = 1, 2, \cdots .$$

Now we solve for F:

$$F_n = b_n e^{-\lambda_n t}$$

for constants b_n. We have therefore found solutions

$$u_n(x,t) = b_n e^{-\lambda_n t} \sin nx.$$

The solution to the PDE plus boundary conditions is then the superposition of all these terms:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

which is the result we found before. The coefficients b_n are found in the same way.

What we have done here is solved for each mode on its own (see homework). Note that if you know the eigenfunctions in advance, you could even guess

$$u_n(x,t) = F_n(t) \sin nx$$

in the first place instead of solving for G, saving even more time.

Note that separation of variables **does not work** for the example in subsection 5.5, because if we plug in $F(t)G(x)$ into

$$u_t = u_{xx} + e^{-t^2} \sin x$$

then the x and t cannot be separated. Thus we need to **identify the eigenfunctions first** and then guess the solution as done in the example.

6.2 The method

Practical note: When it works (for **homogeneous** problems), separation of variables is the easiest way to solve PDEs. Often, the strategy for more complicated problems is to reduce them to simpler ones where separation of variables applies.

For some non-homogeneous problems, eigenfunction expansions are necessary, but even then some separation-of-variables inspired tricks will make the computations easier.

The method of separation of variables is straightforward. Suppose we have some PDE in the variables x and t with boundary conditions. We find the **'general'** solution to the **PDE with the BCs** (leaving initial conditions out) with the following steps:

- Guess a separated solution, a product of functions of each variable:

$$u(x,t) = F(t)G(x).$$

- Plug into the PDE and separate the t's and x's to get

  $$\text{function of t} = \text{function of x}.$$

  Conclude that both are equal to a constant:

  $$\text{function of t} = \text{function of x} = -\lambda.$$

  Note: If this step fails, we must use another method. The negative sign is not needed since λ is arbitrary; it is there to match earlier notation.

- Plug into the boundary conditions to get boundary conditions for one of the ODEs.
- Solve the ODE that has the boundary conditions to get eigenvalues/eigenfunctions. Then solve the other ODE to get solutions $u_n(x,t)$.
- Assume the general solution is an infinite linear combination:

  $$u = \sum_n b_n u_n(x,t).$$

- Solve for the constants using the initial condition for the PDE.

Separation of variables, **when it works**, will produce the correct eigenvalue problem automatically, so it eliminates the need to identify the right operator and reduces the whole process to a concrete series of steps. For instance, consider

$$u_t = u_{xx} + u_x + (t+1)u, \quad u(0,t) = 0, \quad u(1,t) + u_x(1,t) = 0$$

Note that this PDE is linear and homogeneous. Substitute in $u = F(t)G(x)$ to get

$$F'(t)G(x) = F(t)G''(x) + F(t)G'(x) + (t+1)F(t)G(x).$$

Divide by $G(x)$ and $F(t)$ to get:

$$\frac{F'(t)}{F(t)} = \frac{G'' + G'}{G} + (t+1).$$

Subtract $t+1$ and then set both to a constant:

$$\frac{F'(t)}{F(t)} - (t+1) = \frac{G'' + G'}{G} = -\lambda.$$

Now plug into the BCs:

$$F(t)G(0) = 0, \quad F(t)(G(1) + G'(1)) = 0$$

so the eigenvalue problem is

$$G'' + G' = -\lambda G, \quad G(0) = G(1) + G'(1) = 0.$$

Note that this means the operator L from the eigenfunction method is $L[u] = u_{xx} + u_x$ (the $(t+1)u$ part has to be left out). The ODE for F is

$$F' + (\lambda + t + 1)F = 0.$$

With some work, one can then find the eigenvalues/functions λ_n and G_n, then F_n, and get

$$u = \sum_n F_n(t)G_n(x).$$

6.3 Use in solving unfamiliar problems

Separation of variables is particularly useful on more complicated homogeneous problems where the place to start may not be so clear.

Word of caution (which ODE is the eigenvalue problem?): Separation of variables provides a set of ODEs, but we need to solve them in the right order.

In the examples below, note that we need to identify which ODE gives the eigenfunctions and which ODE does not. Doing so requires understanding the boundary and initial conditions, and requires some domain and equation specific intuition.

Wave equation

Consider the **wave equation** in $[0, \pi]$ for $u(x, t)$ with Neumann boundary conditions,

$$u_{tt} = c^2 u_{xx}, \quad u_x(0, t) = u_x(\pi, t) = 0$$

with initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

We substitute

$$u = F(t)G(x)$$

into the PDE to get

$$\frac{1}{c^2}\frac{F''}{F} = \frac{G''}{G} = -\lambda$$

and into the BCs to get

$$F(t)G'(0) = F(t)G'(\pi) = 0 \implies G'(0) = G'(\pi) = 0$$

which gives the eigenvalue problem

$$G'' + \lambda G = 0, \quad G'(0) = G'(\pi) = 0.$$

The other ODE is

$$F'' + c^2 \lambda F = 0.$$

6.3.1 Example:

consider Laplace's equation in a disk of radius a for $u(r, \theta)$, which is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } \theta \in [0, 2\pi],\ r < a$$

with a boundary condition on the boundary of the disk,

$$u(a, \theta) = f(\theta).$$

Assume a separated solution

$$u(r,\theta) = R(r)G(\theta)$$

and substitute into the PDE:

$$R''G + \frac{1}{r}R'G + \frac{1}{r^2}RG'' = 0.$$

Divide by RG to get

$$\frac{R'' + R'/r}{R} + \frac{1}{r^2}\frac{G''}{G} = 0$$

and then move terms around to separate the r and θ; we end up with

$$-r^2\frac{R'' + R'/r}{R} = \frac{G''}{G} = -\lambda.$$

so the ODEs for R and G are

$$G'' + \lambda G = 0, \qquad r^2R'' + rR' - \lambda R = 0$$

Note that plugging into the boundary condition does **not** give us anything since we would need

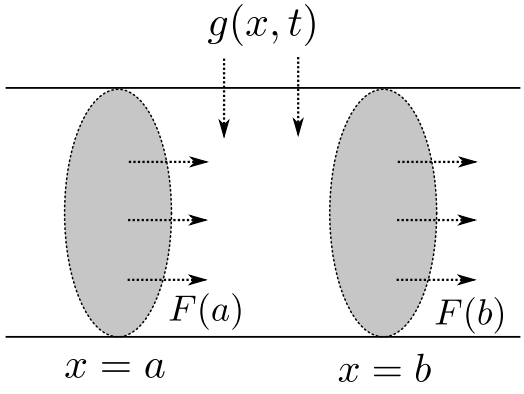
$$R(a)G(\theta) = f(\theta)$$

which is too much to ask for $G(\theta)$ (since $f(\theta)$ will not be a solution to the ODE unless it is an eigenfunction). We will address the missing boundary conditions for G (and R) when solving this equation later.

7 Appendix: additional notes

7.1 Some context: PDEs from conservation laws

Rather than pull the equation out of thin air, let's see how PDEs arise naturally out of fundamental models[4]. To do so, we introduce the concept of a **conservation law**, which is a way of stating that for an amount of stuff in a region, the change in the amount is due to stuff entering/exiting the region or being created/destroyed. For simplicity, assume the stuff is 'heat' - but this argument is quite general (e.g. could be particle concentration, momentum, energy, density of fish, etc.)



[4]Adapted from *Applied Partial Differential Equations*, J. David Logan

Consider a cylindrical tube with cross section A running along the x-direction and $u(x,t)$ the temperature at position x and time t. The amount of heat in a section of the tube for x in some interval $[a,b]$ is

$$\int_a^b u(x,t)\,A\,dx.$$

Let us further suppose there is a **source** $g(x,t)$ that is the rate at which u is created or destroyed at position x along the tube. For instance, heat could leak out of the pipe at a rate $g(x,t)$ if the pipe is poorly insulated.

Define $F(x,t)$ to be the **flux** of heat: the rate at which heat flows through the cross section at x, with units of heat per (area)(time). Thus $\phi A\,dt$ is the amount of heat passing through the cross section in a time dt (with sign determining the direction). We have

$$\underbrace{\frac{\partial}{\partial t}\left(\int_a^b u(x,t)A\,dx\right)}_{\text{change in heat}} = \underbrace{AF(a,t) - AF(b,t)}_{\text{heat entering the section from the ends}} + \underbrace{\int_a^b g(x,t)A\,dx}_{\text{heat created/lost due to source}}.$$

Cancel out A and move the derivative on the LHS inside the integral, leading to

$$\int_a^b u_t(x,t)\,dx = F(a,t) - F(b,t) + \int_a^b g(x,t)\,dx,$$

which is a mathematical description of the conservation of heat.

Now write the F terms in an integral using the Fundamental Theorem of Calculus and collect all the terms to get

$$\int_a^b \left[u_t(x,t) + F_x(x,t) - g(x,t)\right]\,dx = 0.$$

The above equation must hold for **all** intervals $[a,b]$. It follows that the integrand must be equal to zero, leading to the 'differential form' of the conservation law,

$$u_t + F_x = g(x,t).$$

Many models in the sciences arise from this basic conservation argument. The next step is to determine the flux ϕ as as function of u and x (and the source).

7.2 Deriving the heat equation

If u is actually temperature, then the flux can be modeled by **Fourier's law**

$$\phi = -\alpha u_x$$

where α is a constant (the thermal diffusivity, with units of m^2/s). This simple law states that the the flux of heat is towards cooler areas, and the rate is proportional not to the amount of heat but to the gradient in temperature, i.e. the heat will flow faster if there is a large difference (e.g. an ice cube melting in a fridge vs. outside on a hot day).

Thus if there is no external source of heat, then u satisfies the **heat equation**

$$u_t = \alpha u_{xx}.$$

More generally, if u is any quantity whose flux is proportional to minus the gradient of u, then u will also satisfy the above. Such a process is called a **diffusion** process and the equation is then referred to as a **diffusion equation**.