MATH 353 LECTURE NOTES
MORE ON LAPLACE TRANSFORMS

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Topics covered

- Discontinuous forcing
  - Step functions, Laplace transform of steps
  - IVPs with discontinuous forcing
- Convolutions
  - Definition/properties
  - Convolution theorem
  - Transfer function, Laplace vs. time space solutions

1. Introduction (what is the goal?)

A car traveling on a road is, in its simplest form, a mass on a set of springs (the shocks). Bumps on the road apply a force that perturbs the car. A (very) simple model might take the form

$$my'' + by' + ky = F(t)$$

where $y(t)$ is the vertical displacement and $F(t)$ force applied to the system (the 'forcing'). The relation between the input and the response is important for the design of the shocks - to ensure, for instance, that vibrations from bumps are damped out.

In signal processing, filter circuits take an input voltage $V_{\text{in}}(t)$ and output a processed voltage $v(t)$ (e.g. by cutting off high/low frequencies). The input and output might be related by an ODE like[1]

$$av'' + bv' + cv = V_{\text{in}}(t).$$

The goal is to design the circuit so that V_{in} and v are related in the desired way. In ODE terms, this means one needs to **determine the relationship between the inhomogeneous term (the input, or forcing) and the solution (the output).**

We already know from variation of parameters that such a relation exists. However, we are often interested in forcing functions of irregular shapes - such as switches that suddenly turn on, or an impulse applied instantaneously. Moreover, the variation of parameters formula does not give us much to work with.

It will turn out that the Laplace space is a good setting to answer this question, while

[1] not necessarily a real example, but similar; current/voltage in circuits tend to satisfy linear, constant coefficient systems of ODEs.

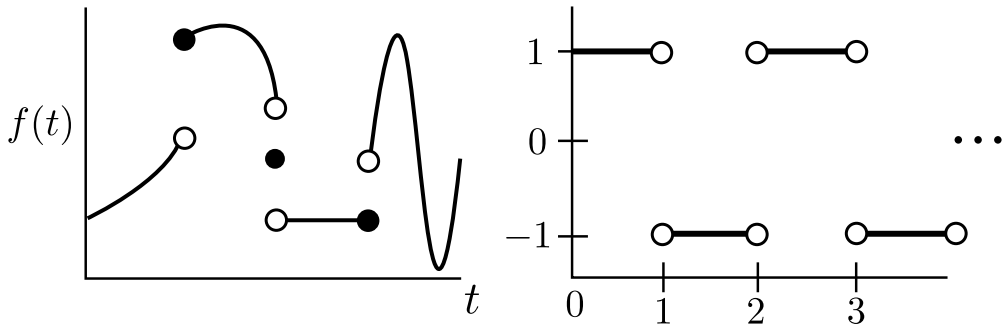


FIGURE 1. Examples of piecewise continuous functions. A generic one (left) and a 'square wave' (right).

the t-space is not.[2] In this section, we will see why.

Often, we want to describe what happens when an input is suddenly turned on, then turned off after some time. A voltage input $V(t)$ that is switched on with value V_0 at $t = 1$, then off at $t = 2$ would correspond to the forcing function

$$V_{\text{in}}(t) = \begin{cases} 0 & t < 1 \\ V_0 & 1 < t < 2 \\ 0 & t > 2 \end{cases}.$$

We need a way to take Laplace transforms of such expressions. The right approach will be to write it a a single formula in terms of a basic function that has a jump.

Remark: A function $f(t)$ is called **piecewise continuous** if it is continuous except at an isolated set of jump discontinuities (see Figure 1). This means that the function is continuous in an interval around each jump. The Laplace transform is defined for such functions (same theorem as before but with 'piecewise' in front of 'continuous'), since

$$\int e^{-st} f(t)\, dt$$

is well-defined if f has jumps. Note that **the value at the jump is irrelevant**, since the integral does not care about values at isolated points.

2. Step functions

The **unit step function** jumps from 0 to 1 at $t = 0$:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases}$$

[2]The *Fourier transform* is the other transform that is used for this purpose; we will not cover it here but will make extensive use of a closely related idea later

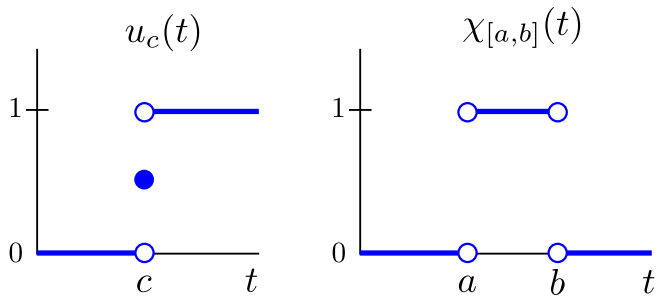
By translation, the jump can be put at any point c:

$$u_c(t) = u(t-c) = \begin{cases} 0 & t < c \\ 1 & t > c. \end{cases} \tag{1}$$

A related function is a 'box' or 'pulse' $\chi_{[a,b]}(t)$ that is 1 in that interval and zero otherwise (formally, called the 'indicator function' of the interval $[a,b]$). In terms of step functions,

$$\chi_{[a,b]}(t) = u_a(t) - u_b(t) = \begin{cases} 1 & a < t < b \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

When an endpoint is at $\pm\infty$, we have $\chi_{(-\infty,a]} = 1 - u_a(t)$ and $\chi_{[b,\infty)} = u_b(t)$.



Notation: The unit step function $u(t)$ is sometimes called the **Heaviside function**. It can be denoted $H(t)$ (`heaviside` in MATLAB), and sometimes other symbols like $\theta(t)$.

By convention, $u(0)$ is taken to be $1/2$ (see figure). However, the value at zero will not be relevant to our discussion.

The function $u_c(t)$ allows us to write **any** piecewise continuous function as a sum of continuous functions times step functions. For instance,

$$g(t) = \begin{cases} \cos t & t < 1 \\ t^2 & t > 1 \end{cases}$$

can be written as

$$g(t) = \cos t + (t^2 - \cos t)u_1(t).$$

This form is a single equation, and it can be Laplace transformed (to be done later).

For multiple discontinuities, one way of writing the function in terms of unit steps is to start with the function before the first discontinuity, then add $(\cdots)u_c(t)$ at each discontinuity c, where $\cdots$ is the difference between the function in the pieces to the right/left of c. As an example, consider

$$g(t) = \begin{cases} t^2 & t < 1 \\ 4t & 1 < t < 2 \\ -5 & 2 < t \end{cases}$$

which has discontinuities at $c = 1, 2$, and 3. The differences in functions for each are $4 - t^2$ at 1, then 9 at 2 (not just the amount **at** c; the difference in *functions* at each piece). Then

$$g(t) = t^2 + (4t - t^2)u_1(t) + (-5 - 4t)u_2(t).$$

Alternate approach: We can use box functions instead, since each one is non-zero exactly in a given interval. We have that

$$\begin{aligned} g(t) &= t^2\chi_{(-\infty,1]} + 4\chi_{[1,2]} - 5\chi_{[2,3]} + e^t\chi_{[3,\infty)} \\ &= t^2(1-u_1) + 4(u_1-u_2) - 5(u_2-u_3) + e^t u_3. \end{aligned}$$

Example (infinite sum) Even if there are infinitely many discontinuities, we can write the function in terms of steps. For instance, the 'square wave' in Figure 1 is

$$f(t) = \begin{cases} 1 & 2n < t < 2n+1 \\ -1 & 2n+1 < t < 2n+2 \end{cases}, \qquad t \geq 0.$$

This function is a 'flat' version of $\sin \pi x$ that goes between -1 and 1. The differences in the function at each discontinuity alternate $-2, 2, -2, \cdots$ so we can write

$$\begin{aligned} f(t) &= u_0(t) - 2u_1(t) + 2u_2(t) - 2u_3(t)\cdots \\ &= u_0(t) + 2\sum_{k=1}^{\infty}(-1)^k u_k(t). \end{aligned}$$

. For each t, the sum has finitely many non-zero terms (why?); the sum converges for all t.

3. Piecewise continuous functions: Laplace transform

The Laplace transform of the step function $u_c(t)$ for $c > 0$ is

$$\mathcal{L}[u_c(t)] = \int_0^\infty e^{-st}u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = \frac{e^{-cs}}{s}, \quad s > 0.$$

If $c < 0$ then $\mathcal{L}$ does not 'see' the discontinuity (because then $u_c = 1$ for $t > 0$).

The step function 'cuts off' the integral below $t < c$ and leaves the rest. More generally, if $f(t)$ is a continuous function and $c > 0$ then

$$u_c(t)f(t-c)$$

is the function that looks like $f(t)$, but starting at $t = c$ instead of $t = 0$ (and is zero before that point). Its Laplace transform is given by

$$\begin{aligned} \mathcal{L}[u_c(t)f(t-c)] &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt \\ &= \int_0^\infty e^{-s(\tau+c)}f(\tau)\,d\tau \qquad (\text{ setting } \tau = t-c) \\ &= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau \\ &= e^{-cs}\mathcal{L}[f(t)]. \end{aligned}$$

The situation is shown in Figure 2.

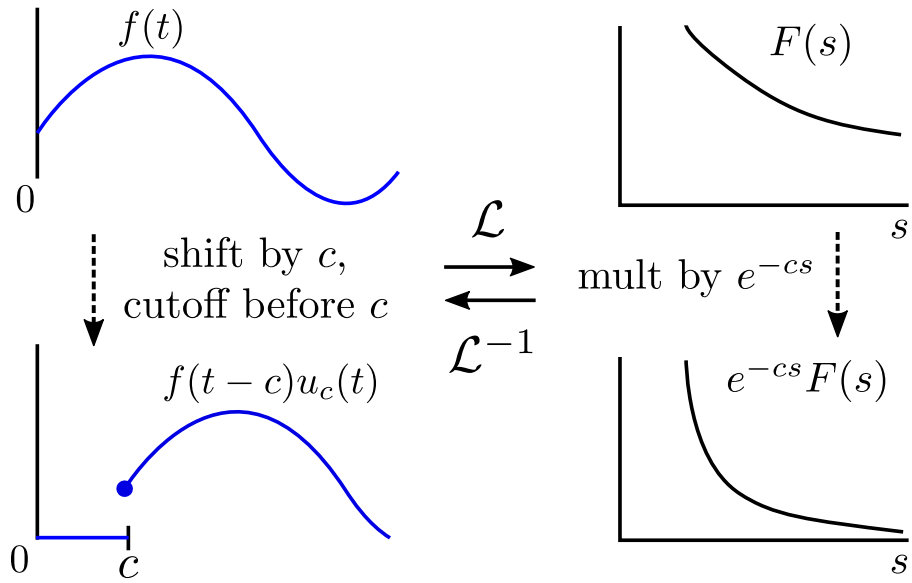


FIGURE 2. Diagram of the step function transform rule. Here 'shift by c' means 'translate the function to the right by c'.

Translation/steps in the t space: If $F(s) = \mathcal{L}[f(t)]$ exists and $c > 0$ then

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s). \tag{3}$$

Informally, this says that translating by c (to the right) in the t-space and cutting off before $t = c$ is equivalent to multiplying by e^{-cs} in the s-space.

If you think of $f(t)$ as 'extended' to be zero for $t < 0$, i.e.

$$f(t) \to f(t)u_0(t) = \begin{cases} \text{some function} & t > 0 \\ 0 & t < 0 \end{cases}$$

then the informal rule is just 'translating by c'. It is safe to do so in our context because the Laplace transform only cares about $t > 0$.

Through the rule (3), it is straightforward to transform functions with jump discontinuities and to solve IVPs with such functions. Write the thing you want to transform in the right form, calculate $F(s) = \mathcal{L}[f(t)]$ separately, then use the formula.

One warning: Be careful about one thing: when using the rule to transform something times $u_c(t)$ you **must write the 'something' in terms of** $t - c$.

For instance, consider

$$\mathcal{L}[\sin(t)u_3(t)].$$

It is **not true that**

$$\mathcal{L}[\sin(t)u_3(t)] = e^{-3s}cL[\sin t] = e^{-3s}/(s^2+1) \text{ (False!)} .$$

The correct $f(t)$ satisfies

$$f(t-3) = \sin t.$$

That is, f shifted by 3 is $\sin t$. To get f, undo this and shift $\sin t$ by -3:

$$f(t) = \sin(t+3).$$

Then the transform is applied to $f(t-3)u_3(t)$:

$$\mathcal{L}[\sin(t)u_3(t)] = e^{-3s}\mathcal{L}[\sin(t+3)].$$

Now use the addition trig. identity and transform term by term to get

$$\mathcal{L}[\sin(t)u_3(t)] = \frac{\cos 3}{s^2+1} + \frac{(\sin 3)s}{s^2+1}.$$

Example with a box: Let

$$f(t) = \begin{cases} 0 & t < \pi \\ \sin(t-\pi) & \pi < t < 2\pi \\ 0 & t > 2\pi \end{cases}.$$

Then $f(t) = \sin(t-\pi)u_\pi - \sin(t-\pi)u_{2\pi}$. Compute term by term:

$$\mathcal{L}[\sin(t-\pi)u_\pi(t)] = e^{-\pi s}\mathcal{L}[\sin t] = \frac{e^{-\pi s}}{1+s^2}.$$

$$\mathcal{L}[\sin(t-\pi)u_{2\pi}] = -\mathcal{L}[\sin(t-2\pi)u_{2\pi}] = -\frac{e^{-2\pi s}}{1+s^2}.$$

Note that the function multiplying $u_{2\pi}$ was rewritten in terms of $(t-2\pi)$ in order to apply the translation rule ($\sin(t-\pi) = -\sin(t-2\pi)$). The transform of $f(t)$ is

$$\mathcal{L}[f(t)] = \frac{e^{-\pi s} + e^{-2\pi s}}{1+s^2}.$$

One more example: Let $f(t) = e^{t-2}$ if $t < 2$ and $f(t) = 1$ if $t > 2$. Then

$$f(t) = e^{t-2} + (1 - e^{t-2})u_2(t).$$

We get

$$\mathcal{L}[f(t)] = e^{-2}\mathcal{L}[e^t] + \mathcal{L}[(1-e^{t-2})u_2(t)] = e^{-2}\mathcal{L}[e^t] + e^{-2s}\mathcal{L}[1-e^t].$$

Now apply the formulas $\mathcal{L}[e^t] = 1/(s-1)$ and $\mathcal{L}[1] = 1/s$ to obtain

$$\mathcal{L}[f(t)] = \frac{e^{-2}}{s-1} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s-1}\right).$$

Translation in s: There is a dual correspondence with translation in the s-space:

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad \text{for } c \in \mathbb{R}.$$

Note that here c is allowed to be negative. Informally:

$$\text{multiplication by } e^{ct} \iff \text{translate by } c.$$

Put another way, translating in the s-space creates exponential factors in t. For instance, given $\mathcal{L}[1] = 1/s$ we can compute

$$\mathcal{L}^{-1}[1/(s-1)] = e^t\mathcal{L}^{-1}[1/s] = e^t.$$

4. IVPs with discontinuous forcing

Now we can solve IVPs with discontinuous forcing terms. The process is the same as before, except that the rule (3) will be used both in taking the transform and the inverse:

1) Transform the ODE, using the transform formula for step functions,
2) End up with $Y(s)$ having terms like $F(s)e^{-cs}$.
3) Break each $F(s)$ into simple pieces.
4) Inverse transform each term, using the step function rule for the e^{-cs} factors.

Step (3) usually involves a partial fraction decomposition. It can be reasonable to do by hand (Example 4.1) but is often tedious (Example 4.2).

4.1. **A standard example.** Consider a mass attached to a damped spring which has displacement $y(t)$ satisfying

$$y'' + 2y' + 2y = f(t), \quad y(0) = y'(0) = 0.$$

The spring is initially at rest, then pulled with force 1 from $t = 1$ to $t = 2$ and let go, i.e.

$$f(t) = u_1(t) - u_2(t).$$

The Laplace transform of f is

$$F(s) = \mathcal{L}[u_1(t)] - \mathcal{L}[u_2(t)] = e^{-s}/s - e^{-2s}/s.$$

Taking the Laplace transform of the IVP, we then find that

$$Y(s) = (e^{-s} - e^{-2s})G(s), \qquad G(s) = \frac{1}{s(s^2 + 2s + 2)}.$$

The inverse transform $g(t) = \mathcal{L}^{-1}[G(s)]$ is obtained by using partial fractions. Write

$$G(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{b + cs}{s^2 + 2s + 2}.$$

Note that $s^2 + 2s + 2$ only has imaginary roots; we could decompose further or leave it as a quadratic. If leaving it, we want it to look like a shift in s of

$$\mathcal{L}[\sin kt] = \frac{k}{s^2 + k^2}, \qquad \mathcal{L}[\cos kt] = \frac{s}{s^2 + k^2}.$$

Solve for a, b, c to get

$$G(s) = \frac{1}{2s} + \frac{-1 - s/2}{s^2 + 2s + 2}.$$

To get the quadratic term right, complete the square:

$$G(s) = \frac{1}{2s} + \frac{-1 - s/2}{(s + 1)^2 + 1}.$$

Now write the numerator in terms of $s + 1$ and split:

$$G(s) = \frac{1}{2s} - \frac{1}{2}\frac{1}{(s + 1)^2 + 1} - \frac{1}{2}\frac{s + 1}{(s + 1)^2 + 1}.$$

A shift by -1 in the s-space ($s \to s + 1$) corresponds to multiplication by e^{-t} in the t-space. The second two fractions are the transforms of sine and cosine shifted by -1, so

$$g(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t$$

The solution is therefore

$$y(t) = u_1(t)g(t-1) - u_2(t)g(t-2).$$

For $t < 1$, $y(t) = 0$ (both terms are 'off'), which makes sense because the system is at rest before the forcing is applied at $t = 1$. In the calculations, this fact is ensured because the e^{-3s} factors appear when taking $\mathcal{L}$ and stick around until we take $\mathcal{L}^{-1}$.

For $t > 2$, both terms are on (the effect of the forcing continues after it stops) and

$$y(t) = \frac{1}{2}e^{-(t-1)}(\sin(t-1) + \cos(t-1) - e\sin(t-2) - e\cos(t-2)).$$

The damping causes the displacement to decay exponentially (while oscillating) after the force stops being applied.

4.2. **Example (not so nice).** We solve the IVP

$$y'' + 3y' + 2y = f(t), \quad y(0) = 1,\ y'(0) = 1,$$

where the forcing is equal to t, switched on at $t = 3$:

$$f(t) = tu_3(t).$$

Note that the initial conditions are non-zero. First, let $F = \mathcal{L}[f]$ and take the transform of the ODE (with $Y = \mathcal{L}[y]$):

$$s^2Y - sy(0) - y'(0) + 3(sY - y(0)) + 2Y = G.$$

Plug in the initial conditions and solve for Y:

$$Y = \underbrace{\frac{s+4}{s^2+3s+2}}_{Y_h} + \underbrace{\frac{F}{s^2+3s+2}}_{Y_p} \tag{4}$$

Homogeneous part: Now solve for the easy part, Y_h (this is the transform of the homogeneous solution; why?). For Y_h, since $s^2 + 3s + 2 = (s+1)(s+2)$ we can write

$$\frac{s+4}{s^2+3s+2} = \frac{a}{s+1} + \frac{b}{s+2}$$

and solve to get

$$Y_h = \frac{3}{s+1} - \frac{2}{s+2}.$$

Inverse transforming, we get

$$y_h = \mathcal{L}^{-1}[Y_h] = 3e^{-t} - 2e^{-2t}. \tag{5}$$

Forcing: Now for Y_p, more calculation is needed. First, we must compute $\mathcal{L}[f]$. To do so, we must write t as $g(t-3)$ for some function g. As noted in the warning on page 5, we get g by shifting t by -3, so

$$F = \mathcal{L}[f] = \mathcal{L}[tu_3(t)] = \mathcal{L}[g(t-3)u_3(t)], \qquad \text{where} g(t) = t + 3.$$

Now by the step function rule and standard transforms,

$$\begin{aligned} F &= \mathcal{L}[f(t-3)u_3(t)] \\ &= e^{-3s}G(s) \\ &= e^{-3s}(1/s^2 + 3/s). \end{aligned}$$

We want to get the inverse transform of Y_p from (**??**), which is

$$Y_p = e^{-3s}F_1(s) \quad \text{where } F_1(s) := \left(\frac{1+3s}{s^2(s+1)(s+2)}\right).$$

The step function rule is used again. Let $f_1 = \mathcal{L}^{-1}(F_1)$. Then

$$y_p = cL^{-1}[Y_p] = f_1(t-3)u_3(t).$$

Collecting all the pieces, the solution to the IVP is

$$y(t) = \underbrace{3e^{-t} - 2e^{-2t}}_{y_h} + \underbrace{f_1(t-3)u_3(t)}_{y_p}.$$

Inverse transform of Y_p: The last thing to do is to actually compute f_1. This is the tedious part. Using partial fractions, we decompose F_1 into a sum of nice parts:

$$F_1 = \frac{3/4}{s} + \frac{1/2}{s^2} - \frac{2}{s+1} + \frac{5/4}{s+2}.$$

Now inverse transform to get

$$f_1(t) = \frac{3}{4} + \frac{1}{2}t - 2e^{-t} + \frac{5}{4}e^{-2t}.$$

Note that the forcing, whose magnitude increases linearly (t), produces a response that also increases linearly (as we expect e.g. from undetermined coefficients).

5. The transfer function

Now we move on to develop more theory and some crucial aspects of the Laplace transform. To motivate the discussion that follows, let us view the ODE

$$y'' + by' + cy = g(t)$$

as describing the response $y(t)$ of a physical system to an input $g(t)$ (recall the examples from section 1.

We are interested in understanding how the output $y(t)$ is related to the input $g(t)$. To simplify matters, lets look at the system starting from rest[3], i.e.

$$y'' + by' + cy = g(t), \quad y(0) = y'(0) = 0. \tag{6}$$

Taking the Laplace transform, we find (by the methods of last week) that there is a function $H(s)$ independent of $g(t)$ such that

$$Y(s) = H(s)G(s)$$

where $Y(s) = \mathcal{L}[y(t)], G(s) = \mathcal{L}[g(t)]$ (check this!). The function $H(s)$ is called the **transfer function**, a key object in analyzing circuits and mechanical systems. For example, the ODE

$$y'' + y = g(t)$$

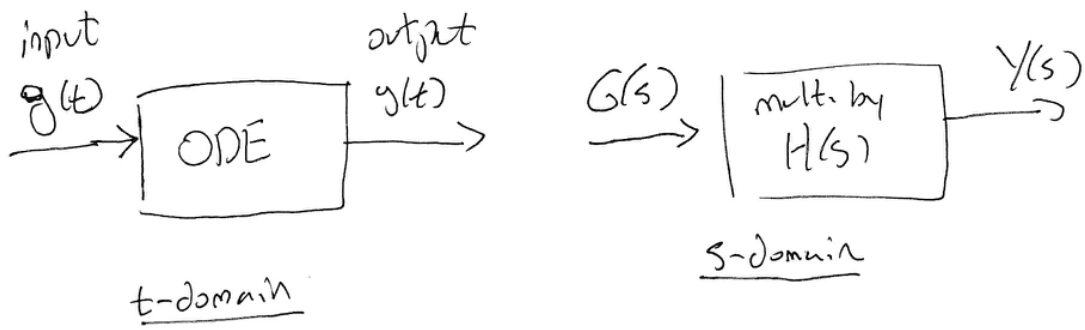
has a transfer function $H(s) = 1/(s^2+1)$.

The transfer function

$$H(s) = \frac{Y(s)}{G(s)}$$

[3]The same arguments apply for higher order constant-coefficient ODEs; we are looking at the second-order case only for simplicity.

is the **ratio of output to input in the Laplace space**, and it is an inherent property of the system. Thus even though $y(t)$ and $g(t)$ might be related in a complicated way, the relation is simple in the Laplace space (s-space). As a diagram:



The system takes the input at a value of s and scales it by $H(s)$. The simplicity of the description makes the s-space a good setting for analysis of such systems - e.g. we can figure out how to dampen unwanted frequencies of a signal by choosing $H(s)$ to be small at the unwanted s-values.

However, one important question remains unanswered - what does 'multiply by $H(s)$' mean back in the original space? That is, how are $y(t)$ and $g(t)$ related?

Recall that using variation of parameters we found that $y'' + y = g(t)$ had a particular solution given by the integral

$$y_p(t) = \int_0^t \sin(t-s)g(s)\,ds.$$

The formula relates y to g, but in a complicated way. It is this expression that we will now study in further detail.

6. Convolutions

The **convolution** of two functions $f(t)$ and $g(t)$ defined on $[0,\infty)$ is

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau. \tag{7}$$

Aside on definition: More generally, for functions f and g defined on $(-\infty,\infty)$, their convolution is defined to be

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau. \tag{8}$$

The above is the form you will usually see as 'the convolution'. We use the definition (7) because the Laplace transform is applied to functions on $[0,\infty)$.

The definition (7) can be seen to be a special case of (8) by extending f and g to be zero for $t < 0$. To be precise, for functions f, g defined on $(-\infty,\infty)$, the formula (7) is the result when applying (8) to $fu_0(t)$ and $gu_0(t)$.

6.1. **Properties.** The convolution obeys some product-like properties:

'Arithmetic' properties of the convolution:

Commutativity: $f * g = g * f$

Associativity: $(f * g) * h = f * (g * h)$

Distributivity: $f * (g + h) = f * g + f * h$

Otherwise, it behaves quite differently from the usual product:

- The value of $(f * g)(t)$ depends not just on $f(t)$ and $g(t)$ but on *all values* of the functions in $[0, t]$ (much more complicated than a product!).
- $f * 1 \neq f$; there is an 'identity' g such that $f * g = f$ but it is not easy to define (we'll see this in looking at impulse functions).
- ...Many other properies we won't need here.

Examples of convolutions: If $f(t) = \sin t$ then

$$f * 1 = \int_0^t \sin(t - \tau)\, ds = \cos(t - \tau)\Big|_0^t = 1 - \cos t.$$

Note that $f * 1$ here denotes $f * g$ where g is the constant function $g(t) = 1$. Sometimes it is easier to swap arguments so that the nicer function has the $t - \tau$ argument:

$$1 * f = \int_0^t \sin(\tau)\, d\tau = -\cos(\tau)\Big|_0^t = 1 - \cos t.$$

If $f(t) = e^t$ then

$$f * f = \int_0^t e^{t-s} e^s\, ds = \int_0^t e^t\, ds = te^t.$$

6.2. **(optional:) Visualization.** It is sometimes useful to give a geometric interpretation of the convolution. We can do so in the following way. Observe that given a value of t, the value of the convolution

$$(f * g)(t) = \int_0^t f(t - \tau) g(\tau)\, d\tau$$

is the (signed) area under the curve $f(t - \tau)g(\tau)$. This curve is obtained by taking $f(-\tau)$, then shifting it right by t and multiplying by $g(\tau)$. Thus we can:

- Extend f, g to be zero for $\tau < 0$.
- Plot $g(-\tau)$ and $f(\tau)$.
- Imagine the graph of $g(-\tau)$ sliding to the right as t increases.
- The area under the product of the two plotted curves is $(f * g)(t)$.

This is a sense in which the convolution is a kind of moving weighted average of two functions (in fact, in can be used directly for smoothing of data). As an example of the visualization, consider $g(\tau) = u_1(\tau) - u_2(\tau)$ (a box of height 1 on $[1, 2]$) and $h(t) = \sin \pi\tau$. Diagrams not included here; see MATLAB code posted elsewhere.

6.3. **Laplace transform.** The Laplace transform of a convolution has a rather remarkable property (see the textbook for the proof):

Convolution theorem (for Laplace transform): Suppose $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$ are defined for $s > a$. Then

$$\mathcal{L}[(f * g)(t)] = F(s)G(s),$$

This establishes a key correspondence:

$$\text{convolutions in } t\text{-space} \iff \text{products in } s\text{-space.}$$

This allows convolutions to be handled via transforms (useful in many problems where convolutions appear), and also lets us take inverse transforms of products.

Immediately, it answers the question of how to relate $y(t)$ to $g(t)$ in the ODE (6). For

$$y'' + by' + cy = g(t), \quad y(0) = y'(0) = 0$$

we know the transform of the solution is

$$Y(s) = H(s)G(s).$$

Now let $h(t) = \mathcal{L}^{-1}[H(s)]$. Then by the convolution theorem,

$$y(t) = \int_0^t h(t-s)g(s)\,ds = (h * g)(t).$$

Thus in the t-space, the output $y(t)$ and input $g(t)$ to the system are related by a convolution of g with a function $h(t)$ depending on the ODE.

6.4. **Non-zero initial conditions.** Suppose now that $y(0) = y_0$ and $y'(0) = y_0'$. Then

$$Y(s) = H(s)(G(s) + sy_0 + y_0')$$

In this case we obtain the solution in terms of particular/homogeneous solutions as follows:

$$y(t) = \underbrace{(h * g)(t)}_{y_p(t)} + \underbrace{\mathcal{L}^{-1}[H(s)(sy_0 + y_0')]}_{y_h(t)}.$$

One might be tempted to evaluate the second term as a convolution as well; but that would require taking $\mathcal{L}^{-1}[1]$ and $\mathcal{L}^{-1}[s]$. **It is important to note that** with what we know so far, the formula

$$\mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]$$

only applies when F and G are the Laplace transforms of functions f and g. Unfortunately, there is no function $f(t)$ such that

$$\mathcal{L}[f] = 1.$$

To get around the issue, we will need to do more work - the subject of the next section.

6.5. **Using the convolution theorem.** The convolution theorem is obviously useful for computing the transform of a convolution. It is also useful for computing inverse transforms of products (sometimes). Some examples:

$\mathcal{L}^{-1}$ using the convolution theorem:

Example 1: Let's find the inverse transform of

$$F(s) = \frac{1}{s^3(s^2+1)}.$$

We write F as a product of easier-to-invert functions and then inverse transform it using the convolution theorem. Let $G(s) = 1/s^3$ and $H(s) = 1/(s^2+1)$ and let $f = \mathcal{L}^{-1}[G], h = \mathcal{L}^{-1}[H]$. Then

$$\mathcal{L}^{-1}[F(s)] = (f * g)(t).$$

Both G and H are standard transforms; we find that $g(t) = t^2/2$ and $h(t) = \sin t$. From here, either declare victory or evaluate the convolution integral explicitly (using some integration by parts):

$$\begin{aligned}
\mathcal{L}^{-1}[F(s)] &= \int_0^t f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^t \frac{1}{2}(t-\tau)^2 \sin(\tau)\,d\tau \\
&= -\frac{1}{2}(t-\tau)^2\cos\tau\Big|_0^t + \int_0^t (t-\tau)\cos(\tau)\,d\tau \\
&= -\frac{1}{2}t^2 + (t-\tau)\sin(\tau)\Big|_0^t - \int_0^t \sin(\tau)\,d\tau \\
&= -\frac{1}{2}t^2 + \cos t - 1.
\end{aligned}$$

Note that this approach is about the same amount of work as using partial fractions.

Example 2: It may be tempting to try to find $\mathcal{L}^{-1}$ of

$$F(s) = \frac{s}{s^2+1}$$

using the convolution theorem, by setting $G(s) = s$ and $H(s) = 1/(s^2+1)$. Then

$$\mathcal{L}^{-1}[F(s)] = \int_0^t g(t-s)\sin(\tau)\,d\tau$$

where $g(t) = \mathcal{L}^{-1}[G(s)]$. The problem is that $\mathcal{L}[\delta'] = s\mathcal{L}[\delta] = s$ so g is not a function. One can get the right answer by substituting in $g = \delta'$ but that requires some care (and technical issues!).

The easier approach is either to just recognize this is $\cos t$ or to use partial fractions to write it as $a/(s-i) + b/(s+i)$.