# MATH 353 LECTURE NOTES
# LAPLACE TRANSFORM: FUNDAMENTALS

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## Topics covered

- Introduction to the Laplace transform
- Theory and definitions
  - Domain and range of $\mathcal{L}$
  - Inverse transform
  - Fundamental properties
    - linearity
    - transform of derivatives
- Use in practice
  - Standard transforms
  - A few transform rules
  - Using $\mathcal{L}$ to solve constant-coefficient, linear IVPs
  - Some basic examples

## 1. The idea

We turn our attention now to *transform methods*, which will provide not just a tool for obtaining solutions, but a framework for understanding the structure of linear ODEs.

The idea is to define a transform operator $\mathcal{L}$ on functions,

$$\mathcal{L} : \text{origin space} \to \text{transformed space}$$

such that the ODE in the transformed space is much easier to solve. We will consider an **integral transform**, which takes the form

$$\mathcal{L}[f(t)] = \int_D K(s,t) f(t)\, dt$$

where $D$ is some domain (usually $(-\infty,\infty)$ or $(0,\infty)$) and $K(s,t)$ is a function called the **kernel** of the transform. One of the two most important integral transforms[1] is the **Laplace transform** $\mathcal{L}$, which is defined according to the formula

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

i.e. $\mathcal{L}$ takes a function $f(t)$ as an input and outputs the function $F(s)$ as defined above.

[1] the other is the Fourier transform; we'll see a version of it later.

$$\begin{tikzcd}
\text{origin space} & \text{ODE in } t \arrow[r, "\mathcal{L}"] \arrow[d, dotted] & \text{eqn. in } s \arrow[d] & \text{transform space} \\
& y(t) & Y(s) \arrow[l, "\mathcal{L}^{-1}"] &
\end{tikzcd}$$

They key properties of the Laplace transform (which we'll look at in detail) are:

- $\mathcal{L}$ is a linear operator
- $\mathcal{L}$ turns differentiation in $t$ into *multiplication* by $s$ (almost):
$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0).$$
- $\mathcal{L}^{-1}$ exists and both $\mathcal{L}$ and $\mathcal{L}^{-1}$ can be computed in practice

Because of these properties, given an ODE
$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$$
we can:

1) Use the transform to convert this into an **algebraic** equation for $Y = \mathcal{L}[y]$.
2) solve for $Y$ in the $s$-space
3) Apply $\mathcal{L}^{-1}$ to return to the $t$-space and get $y(t)$

Typically, the equations in (2) are much easier to work with than the ODE. The $s$-space will tell us information about the solution that would be difficult to obtain directly.

**motivating example** For example, let's solve
$$0 = y' - y$$
Let $Y(s) = \mathcal{L}[y(t)]$ be the Laplace transform of the solution. Applying $\mathcal{L}$ to the equation, we obtain the transformed equation
$$\mathcal{L}[0] = \mathcal{L}[y'] - \mathcal{L}[y] = sY - y(0) - Y.$$
Since $\mathcal{L}[0] = 0$, we get
$$0 = (s-1)Y - y(0),$$
which is trivial to solve! The transformed solution to the ODE is then
$$Y(s) = \frac{y(0)}{s-1}.$$
Here is the point at which we have to do actual work - the price of transforming the ODE is that we have to undo the transformation to get the desired solution, $y(t)$. In this case, it is easy to show that $\mathcal{L}[e^t] = 1/(s-1)$, from which we can conclude that
$$y(t) = y(0)e^t.$$

## 2. The laplace transform

Now we go through the basic theory. The treatment here is not be completely rigorous; some technical details are omitted in favor of getting to the key points.

**Definition:** The **Laplace transform** $F(s)$ of a function $f(t)$ is

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{2}$$

defined for all $s$ such that the integral converges.

2.1. **Domain/range of the Laplace transform.** We want to find a set of functions for which (2) is defined for large enough $s$. For (2) to be defined, we need that:

- $f$ is integrable and defined for $[0, \infty)$
- $f$ grows more slowly than the $e^{-st}$ term

Hereafter, **we shall assume that** $f$ is defined on the domain $[0, \infty)$ unless otherwise noted.

**Definition:** A function $f(t)$ is **piecewise continuous** if it is continuous except for an isolated set of jump discontinuities.[2]

**Definition:** A function $f(t)$ is of **exponential type** (or **order**) if there are constants $a$ and $k$ such that

$$|f(t)| \le K e^{at}. \tag{3}$$

**Note:** Technically, this need only hold as $t \to \infty$, but the distinction is not important here.

These two properties are enough to guarantee $\mathcal{L}$ is defined:

**Theorem:** If $f$ is piecewise continuous and of exponential type as in (3) then

$$\mathcal{L}[f(t)] = F(s) \text{ is defined for all } s > a.$$

Informally: If $f$ grows slower than $e^{st}$ then $F(s)$ is defined, so if $f$ grows slower than $e^{at}$ for some $a$ then $F(s)$ is defined for all $s > a$.

*Proof.* (Sketch.) We need to show that $\int_0^\infty e^{-st} f(t)\, dt$ is finite when $s > a$. Use the bound on $f$ to estimate

$$\left| \int_0^\infty e^{-st} f(t)\, dt \right| \le \int_0^\infty K e^{-(s-a)t}\, dt = \frac{K}{s-a}$$

which is finite so long as $s > a$. It follows (omitting technical details) that the integral exists and is finite, so $\mathcal{L}$ is defined for $s > a$. □

**Note on the theorem and proof:** The condition (3) can be replaced with the weaker condition that $|f(t)| \le K e^{at}$ for $t > M$ for some $M$ (that is, $f$ is eventually bounded by an exponential). It does not matter what $f$ does in a finite interval, which allows the assumptions to be relaxed a bit.

For the proof, there is a problem since $\int_0^\infty e^{-st} f(t)\, dt$ is not known to exist in the first place. To be correct, we must use a comparison test for integrals: If there is an 'upper bound' function $h(t)$ such that

$$|g(t)| \leq h(t) \text{ and } \int_0^\infty h(t)\,dt < \infty$$

then $\int_0^\infty g(t)\,dt$ exists (and is finite).

## 3. Fundamental properties

The most basic property is also the most essential, so it gets a box:

**Linearity:** The Laplace transform $\mathcal{L}$ is a linear operator.

**Proof:** Suppose $f_1, f_2$ are functions for which $\mathcal{L}$ is defined and $c_1, c_2 \in \mathbb{R}$. Then

$$\begin{aligned}\mathcal{L}[c_1 f_1 + c_2 f_2] &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt \\ &= c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2].\end{aligned}$$

Note that if $\mathcal{L}[f_1]$ and $\mathcal{L}[f_2]$ are defined for $s > a$ then the same is true of the linear combination $c_1 f_1 + c_2 f_2$.

The other key property is that it acts in a nice way on derivatives:

**Theorem:** Suppose $f$ is of exponential type (i.e. (3) holds) and $f'$ is piecewise continuous. Then $\mathcal{L}[f'(t)]$ exists on the same domain as $\mathcal{L}[f]$ and

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0). \tag{4}$$

Conceptually, it is essential to understand to that the above means

$$\text{derivatives in the original space} \iff \text{multiplication in the transformed space}$$

up to the extra terms. The proof is straightforward and worth knowing. For simplicity, assume that $f$ is continuous. Let $a$ be the constant in (3); that is,

$$|f(t)| \leq Ke^{at}.$$

To get the formula, integrate by parts (carefully):

$$\begin{aligned}\int_0^\infty e^{-st} f'(t)\,dt &= \lim_{b\to\infty} e^{-bt} f(t) - f(0) - \lim_{b\to\infty} \int_0^b (-se^{-st}) f(t)\,dt \\ &= -f(0) + s \lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt \\ &= -f(0) + s\mathcal{L}[f(t)].\end{aligned}$$

The first limit is zero by the bound on $f$ since

$$|e^{-st}f(t)| \leq Ke^{-(s-a)t} \to 0 \text{ as } t \to \infty.$$

For the second limit, we can take $b \to \infty$ since we have already established the improper integral converges in the proof that $\mathcal{L}[f]$ exists.

This result can be iterated to find the Laplace transform of higher order derivatives. For example,

$$\begin{aligned} \mathcal{L}[f''(t)] &= s\mathcal{L}[f'(t)] - f'(0)) \\ &= s(s\mathcal{L}[f(t)] - f(0)) - f'(0) \\ &= s^2\mathcal{L}[f(t)] - sf(0) - f'(0) \end{aligned}$$

and so on. Thus an $n$-th derivative in the original space correspond to multiplications by $s^n$ in the transformed space (up to some polynomial in $s$). To be precise, we have:

**Theorem (Laplace transform of derivatives):** If $f^{(n)}$ is piecewise continuous and $f$ and all its derivatives up to $n-1$ are of exponential type then

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{5}$$

As before, if the transforms of $f, f', \cdots, f^{(n-1)}$ are defined for $s > a$ then the transform of $f^{(n)}$ is also defined for $s > a$.

3.1. **Inversion.** The Laplace transform has an inverse; for any reasonable nice function $F(s)$ there is a unique $f$ such that $\mathcal{L}[f] = F$.

**Inverse of the Laplace transform:** If $F(s)$ is defined for $s > a$ then there is a unique function $f(t)$ such that

$$\mathcal{L}[f(t)] = F(s).$$

In this case we write

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

Unfortunately, the details (and definition of $\mathcal{L}^{-1}$ in general) require some complex analysis and are beyond the scope of this course. The inverse is notoriously difficult to work with in general. In practice, one typically computes $\mathcal{L}^{-1}[F(s)]$ by recognizing $F(s)$ as comprised of known transforms. In the next section, we derive some 'standard' transforms; these functions (along with some other known results) will be the things whose inverses are known.

## 4. Inverses and transforms

In this section we compute some common transforms and show strategies for computing the inverse transform of a function $F(s)$. This discussion will involve deriving some new properties of $\mathcal{L}$ and will make use of a few results from calculus.

First, let us derive/state some basic transforms.

4.1. **Easy cases.** For some functions $\mathcal{L}$ can be computed by integrating directly.

**Constant function:** For the constant function $f(t) = 1$,

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \text{ defined for } s > 0.$$

**Exponential:**

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad s > a.$$

**Sine/cosine:** The formula above applies for complex exponentials. In particular,

$$\mathcal{L}[e^{(a+bi)t}] = \frac{1}{s-(a+bi)} = \frac{s-a+bi}{(s-a)^2+b^2}.$$

In particular, because $\mathcal{L}$ is linear we can take real and imaginary parts to get

$$\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}, \quad \mathcal{L}[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}.$$

**Polynomial:** Let $n \geq 1$ be an integer. Then, using integration by parts, we can find the transform of $t^n$ in terms of the transform for $t^{n-1}$. If $s > 0$ then

$$\begin{aligned}\mathcal{L}[t^n] &= \int_0^\infty t^n e^{-st}\,dt \\ &= \frac{-1}{s} t^n e^{-st}\Big|_{t=0}^\infty - \int_0^\infty n t^{n-1}\left(-\frac{1}{s}e^{-st}\right)dt \\ &= \frac{n}{s}\mathcal{L}[t^{n-1}].\end{aligned}$$

Note that both of the boundary terms in the integration by parts are zero ($t^n e^{-st} \to 0$ as $t \to \infty$ and is equal to zero at $t = 0$). Iterating $n$ times and using $\mathcal{L}[t^0] = 1/s$, we find that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

**Non-integer case:** The transform for $t^p$ when $p$ is a real number is not as nice. Define the **gamma function**

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\,dt, \qquad p > 0.$$

Note that $\Gamma(n) = (n-1)!$ if $n$ is an integer. If $p > -1$ then

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}.$$

## 5. Transform rules/equivalences

Not every $F(s)$ is going to be immediately recognizable as a standard transform. Often, we need to break it into manageable parts. If you see something like

$$\frac{s^2+3}{(s-1)^2(2s-2)^2}$$

you should think: how can this be turned into a sum of easy-to-invert functions? There are a number of rules to break expressions down.

Many of the rules are really correspondence between operations in the original space and the transform space (like differentiation in $t$ being multiplication by $s$). They can be used to compute $\mathcal{L}$ or $\mathcal{L}^{-1}$, but are also useful for analysis.

The list will grow considerably as the discussion progresses!

**Linearity:** The inverse transform $\mathcal{L}^{-1}$ is linear. Thus sums can be inverted term by term and constant factors can be moved in/out of the transform.

$$\mathcal{L}^{-1}[cF(s)] = c\mathcal{L}^{-1}[F(s)],$$
$$\mathcal{L}^{-1}[F_1(s)+\cdots+F_n(s)] = \mathcal{L}^{-1}[F_1(s)]+\cdots+\mathcal{L}^{-1}[F_n(s)].$$

**Derivatives in** $s$: A dual property to the rule for $f'(t)$. A derivative in the transformed space corresponds to multiplication by $(-t)$ in the original space:

$$(-t)^n f(t) \rightleftharpoons F^{(n)}(s)$$

For example,

$$\mathcal{L}[te^t] = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}.$$

This could also be used in reverse to find $\mathcal{L}^{-1}[1/(s-1)^2]$.

**Scaling** $s$ **and** $t$**:** The argument to the function $F(s)$ or $f(t)$ can be scaled by a constant (positive) factor. If $c>0$ is a constant then

$$f(ct) \rightleftharpoons \frac{1}{c}F(s/c).$$

For example, to invert $e^{-as}$, we can calculate

$$\mathcal{L}^{-1}[\frac{a}{s^2+a^2}] = \frac{1}{a}\mathcal{L}^{-1}[\frac{1}{(s/a)^2+1}] = \sin(at)$$

since $\frac{1}{(s/a)^2+1)}$ is $F(s/a)$ where $F=\mathcal{L}[\sin t]$.

The first three are the most commonly used. There are a few others to be derived later.

## 6. Solving ODEs with the Laplace transform

We are now ready to use the Laplace transform to solve linear, constant coefficient initial value problems, that is equations of the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f$$

where the $a_i$'s are constants.

**Remark:** Example 2 (below) shows that the transform has no problem with inhomogeneous term, so long as we can transform/inverse transform them. Any order is also fine (see Example 3), but it makes the calculations much more involved.

The procedure is as follows:

1) Apply $\mathcal{L}$ to the ODE to obtain an equation for $Y(s) = \mathcal{L}[y(t)]$. Use the initial conditions to evaluate the $y(0), y'(0)$ etc. terms in (5). (*easy; always the same process*)
2) Solve the equation for $Y(s)$ (*easy*)
3) Decompose $Y(s)$ into a sum of functions that are easy to invert. (*can be tricky depending on the form of $Y(s)$; it can be a mess*).
4) Calculate each term of $\mathcal{L}^{-1}[Y(s)]$ (*mostly straightforward if Step 3 was done well*)

$$\begin{tikzcd} t\text{-space} & & s\text{-space} \\ y^{(n)}+\cdots=f(t) \arrow[rr, "\mathcal{L}"] & & s^nY(s)+\cdots=F(s) \arrow[d, "\text{solve for } Y"'] \\ & & Y(s)=\cdots \arrow[d, "\text{decompose}"'] \\ y(t)=\cdots & & Y=Y_1+Y_2+\cdots \arrow[ll, "\mathcal{L}^{-1}"] \end{tikzcd}$$

It is worth noting, and we will see later, that the **end goal is not always to get a formula for** $y(t)$. If we are interested in understanding the behavior of solutions, the transform space can be the right place to do analysis.

**Example 1 (homogeneous):** A simple initial value problem.

$$y'' - 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Let $Y(s) = \mathcal{L}[y(t)]$. Take the transform of the ODE, then apply the initial conditions:

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) - 2(sY - y(0)) + Y \\ &= s^2Y - s - 2sY + 2 + Y \\ &= (s^2 - 2s + 1)Y - s + 2 \\ &= (s-1)^2 Y - s + 2. \end{aligned}$$

The solution in the transformed space is therefore

$$Y(s) = \frac{s-2}{(s-1)^2}.$$

Now we write (this is an example of partial fractions)

$$Y(s) = \frac{s-2}{(s-1)^2} = \frac{1}{s-1} - \frac{1}{(s-1)^2}.$$

The first term is $\mathcal{L}[e^t]$; for the second, see below. Inverting, we get

$$\begin{aligned}\mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}[\frac{1}{s-1}] + \mathcal{L}^{-1}[\frac{1}{(s-1)^2}] \\ &= e^t + (-t)e^t \\ &= (1-t)e^t.\end{aligned}$$

*For the second term:* Use the derivative-in-$s$ rule, observing that

$$-\frac{1}{(s-1)^2} = \frac{d}{ds}\left(\frac{1}{s-1}\right). \tag{6}$$

We know that $d/ds$ corresponds to multiplication by $-t$, i.e.

$$\mathcal{L}[(-t)^n f(t)] = F^{(n)}(s)$$

so we can use this to take the inverse transform of (6) to get

$$\mathcal{L}^{-1}[1/(s-1)^2] = (-t)e^t.$$

*Alternate method:* Use the translation in $s$ rule,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c)$$

and the fact that $\mathcal{L}[t] = 1/s^2$ to get

$$\mathcal{L}^{-1}[\frac{1}{(s-1)^2}] = e^t cL^{-1}[\frac{1}{s^2}] = te^t.$$

**Example 2 (inhomogeneous):** An initial value problem with a forcing term.

$$y'' + y = \sin\omega t, \quad y(0) = 0, y'(0) = 1, \qquad \omega \neq \pm 1.$$

Take the transform (using the standard result for sine ):

$$s^2Y - sy(0) - y'(0) + Y = \frac{\omega}{s^2+\omega^2}.$$

Apply the initial condition and obtain

$$(s^2+1)Y = 1 + \frac{\omega}{s^2+\omega^2}.$$

so we get

$$Y = \frac{1}{s^2+1} + \frac{\omega}{(s^2+1)(s^2+\omega^2)}.$$

Now use partial fractions:

$$\frac{1}{(s^2+1)(s^2+\omega^2)} = \frac{A}{s^2+1} + \frac{B}{s^2+\omega^2}$$

which gives

$$1 = As^2 + A\omega^2 + Bs^2 + B$$

so $A + B = 0$ and $A\omega^2 + B = 1$, solved by $A = 1/(\omega^2 - 1)$ and $B = -1/(\omega^2 - 1)$. Thus $Y$, after using partial fractions, is

$$Y = \frac{1}{s^2+1} + \frac{\omega}{(\omega^2-1)}\left(\frac{1}{s^2+1} - \frac{1}{s^2+\omega^2}\right).$$

Now we recognize that

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}, \quad \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2},$$

and use the known transforms to invert each term of $Y$:

$$Y = \frac{\omega^2+\omega-1}{\omega^2-1}\sin t - \frac{1}{\omega^2-1}\sin \omega t.$$

**Example 3 (higher order ODE):** A fourth order IVP. The method works for any linear constant-coefficient ODE (of any order). We solve

$$y^{(4)} - 5y'' + 4y = 0, \qquad y(0) = 1,\ y'(0) = 0,\ y''(0) = 3,\ y'''(0) = 0.$$

Take the Laplace transform and use the initial conditions:

$$(s^4Y - s^3 - 3s) - 5(s^2Y - s) + 4Y = 0.$$

Solve for $Y$ to obtain

$$Y(s) = \frac{s^3-2s}{s^4-5s^2+1} = \frac{s^3-2s}{(s-1)(s+1)(s-2)(s+2)}.$$

Notice that the denominator just the characteristic polynomial of the ODE (which will be true in general). We can invert $Y(s)$ by using partial fractions to write it in the form

$$Y(s) = \frac{1/6}{s-1} + \frac{1/6}{s+1} + \frac{1/3}{s-2} + \frac{1/3}{s+2}.$$

(The calculation here is tedious but straightforward). Each term can be inverted using $\mathcal{L}[e^{at}] = 1/(s-a)$ to obtain the solution

$$y(t) = \frac{1}{6}(e^t + e^{-t}) + \frac{1}{3}(e^{2t} + e^{-2t}).$$