1. Motivation

1.1. Linear algebra analogy. Suppose we have an $n \times n$ matrix $A$ and wish to solve

$$Ax = b.$$ 

Assume $A$ is invertible and symmetric. You know many ways of solving this problem, but let’s go through the details of one approach that will be a useful analogy. Rather than solve a linear system, we would like to choose the right basis so that the $n$ equations to solve are independent.

To do so, recall that an $n \times n$ real symmetric matrix has distinct eigenvalues

$$\lambda_1, \lambda_2, \cdots, \lambda_n$$

and eigenvectors

$$v_1, v_2, \cdots, v_n$$

that form a basis for $\mathbb{R}^n$. Not only that, but they are orthogonal:

$$v_i \cdot v_j = 0 \quad \text{if } i \neq j.$$
Now both $x$ and $b$ can be written in terms of this basis:

$$x = \sum_{i=1}^{n} c_i v_i, \quad b = \sum_{i=1}^{n} d_i v_i.$$  

Now since $A v_i - \lambda_i v_i$, plugging into $Ax = b$ gives

$$\sum_{i=1}^{n} \lambda_i c_i v_i = \sum_{i=1}^{n} d_i v_i.$$  

Since the $v_i$’s form a basis, it must be true that the coefficients of each $v_i$ are equal, so

$$c_i = d_i / \lambda_i.$$  

This leaves only the problem of writing $b$ in terms of the $v_i$’s. Here is where orthogonality is crucial. Take the dot product of the equation for $b$,

$$b = \sum_{i=1}^{n} d_i v_i,$$  

with $v_j$ to get

$$b \cdot v_j = \sum_{i=1}^{n} d_i (v_i \cdot v_j).$$  

Now all the terms in the sum are zero except for $i = j$:

$$b \cdot v_j = d_i v_j \cdot v_j \implies d_i = \frac{b \cdot v_j}{v_j \cdot v_j}.$$  

The solution to $Ax = b$ is then

$$x = \sum_{i=1}^{n} c_i v_i, \quad c_i = \frac{1}{\lambda_i} \frac{b \cdot v_i}{v_i \cdot v_i}.$$  

Assuming we have the eigenvectors and eigenvalues, notice that at no point did we need to solve any coupled linear systems. All the equations (for $d_i$ and then $c_i$) only had one term divided by another term. The orthogonality of the basis allowed us to decompose $b$ in terms of the basis by solving for the $d_i$’s one by one, and the eigenvalue property $A v = \lambda v$ let us solve for $c_i$ in terms of $d_i$. In the coming weeks, we will see how to generalize these ideas to differential equations.

1.2. Heat conduction in a metal bar. A metal bar with length $L = \pi$ is initially heated to a temperature of $u_0(x)$. The temperature distribution in the bar is $u(x, t)$. Over time, we expect the heat to diffuse or be lost to the environment until the bar is evenly heated.
Physicist Joseph Fourier, around 1800, studied this problem and in doing so drew attention to a novel technique that has since become one of the cornerstones of applied mathematics. The approach outlined below hints at some of the deep structure we will uncover in the remainder of the course.

We will show later that the temperature can be modeled by the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0 \text{ and } x \in (0, \pi).$$

Assume that the temperature is held fixed at both ends. This condition is imposed in the model through boundary conditions

$$u(0, t) = u(\pi, t) = 0 \text{ for all } t.$$

Notice that unlike initial conditions, there are boundary conditions at two different values of $x$. Thus we do not have an ‘initial value problem’ where we start at one point.

Lastly, the initial heat distribution is $t = 0$ is

$$u(x, 0) = u_0(x)$$

which is the initial condition. The temperature should decrease as heat leaks out of the bar through the ends; eventually it all dissipates. The solution $u(x, t)$ should predict this.

In summary, our goal is to find a function $u(x, t)$ defined on $[0, \pi]$ satisfying

(1a) \hspace{1cm} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0 \text{ and } x \in (0, \pi),

(1b) \hspace{1cm} u(0, t) = u(\pi, t) = 0 \text{ for } t \geq 0

(1c) \hspace{1cm} u(x, 0) = u_0(x).

We call (1) (the three equations together) an initial boundary value problem for $u(x, t)$.

To solve the equation, we guess at an exponentially decaying solution

(2) \hspace{1cm} u(x, t) = e^{-\lambda t} \phi(x).

Our objective here is just to find a solution to the first two parts, (1a) and (1b) and worry about the initial condition later.

Substituting into the PDE (1a), we find that

$$-\lambda \phi(x) = \phi''(x).$$
Now substitute into the boundary conditions (1b) (note that $e^{-\lambda t}$ cancels out here) to get
\[ \phi(0) = 0, \quad \phi(\pi) = 0. \]
For convenience set $\lambda = \mu^2$. It follows that (2) satisfies the PDE (1a) and the boundary conditions (1b) if the function $g(x)$ solves the boundary value problem
\[ \phi''(x) + \mu^2 \phi(x) = 0, \quad \phi(0) = 0, \phi(\pi) = 0. \]
This problem is not an initial value problem, but it is a constant-coefficient ODE, so we can still solve it explicitly. The general solution is
\[ \phi = c_1 \sin(\mu x) + c_2 \cos(\mu x). \]
Imposing the condition $\phi(0) = 0$ we find that
\[ \phi = c_1 \sin(\mu x). \]
The second condition, $\phi(1) = 0$, requires that
\[ \sin(\mu \pi) = 0. \]
Non-trivial solutions exist whenever $\mu$ is a non-zero integer. We have now found an infinite sequence of solutions to (3):
\[ \phi_n(x) = \sin(nx), \quad n = 1, 2, 3, \ldots \]
Observe that (3) is a linear, homogeneous problem. In particular,
\[ \phi_1, \phi_2 \text{ are solutions to (3)} \implies c_1 \phi + c_2 \phi_2 \text{ is a solution}. \]
This means that for any constant $a_n$, the function
\[ a_n e^{-n^2 t} \phi_n(x) \]
is a solution to the heat conduction problem with initial data
\[ u_0(x) = a_n \sin(nx). \]
Now the crucial question: what happens when the initial data is not a sine? No single solution of the form (5) will work. Fourier’s breakthrough was the realization that, using the superposition principle (4), the solution could be written as an infinite linear combination of all the solutions of the form (5):
\[ u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \phi_n(x). \]
Then $u(x, t)$ solves the original problem (1) if the coefficients $a_n$ satisfy
\[ u_0(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \]
This idea is a generalization of what you know from linear algebra (representing vectors in terms of a basis) but with basis functions \{\sin(nx) : n = 1, 2, 3, \ldots\}.
In fact, this set of functions has the rather remarkable orthogonality property
\[ \int_0^\pi \phi_m(x) \phi_n(x) \, dx = \int_0^\pi \sin(mx) \sin(nx) \, dx = 0, \quad m \neq n. \]
To solve for the coefficient $a_m$, we can multiply (6) by $\sin(mx)$ and integrate:

$$\int_0^\pi u_0(x) \sin(mx) \, dx = \int_0^\pi \sum_{n=1}^\infty a_n \sin(mx) \sin(nx) \, dx.$$  

Now move the integral inside the sum (it is not trivial to show this is allowed!). By the property (7), only one of the terms in the sum will be non-zero:

$$\int_0^\pi u_0(x) \sin(mx) \, dx = \sum_{n=1}^\infty a_n \int_0^\pi \sin(mx) \sin(nx) \, dx$$

$$= \left( \sum_{n=1, n \neq m}^\infty a_n \cdot 0 \right) + a_m \int_0^\pi \sin(mx) \sin(mx) \, dx$$

$$= a_m \int_0^\pi \sin^2(mx) \, dx.$$  

Magically, the infinite sum has been reduced to a simple equation for $a_m$:

$$a_m = \frac{\int_0^\pi u_0(x) \sin(mx) \, dx}{\int_0^\pi \sin^2(mx) \, dx}. \quad (8)$$

This process works for all $m$, so the solution to the heat conduction problem (5) with arbitrary initial condition $u_0(x)$ is

$$u(x,t) = \sum_{n=1}^\infty a_n e^{-n^2t} \sin(nx)$$

with the coefficients given by the formula (8). Of course, all of the manipulations here are formal and unjustified - it is far from clear whether the series converges, or if it is valid to swap integrals and sums, and so on (Fourier did not know this either when first applying the method; it took several decades to settle the issue).

1.3. Observations and goals. The method in the example may seem rather mysterious, but it hints at some remarkable structure. We have identified eigenfunctions $\phi_n$ that satisfy

$$\phi_n'' = \lambda \phi_n, \quad \phi_n(0) = \phi_n(\pi) = 0$$

and found that they have a special orthogonality property. Then, we exploited superposition to build an infinite series, which has enough coefficients to match any initial condition. The functions $\phi_n$ must span all possible initial conditions - they are a basis in some sense. While building up the theory, we need to address some fundamental questions first:

- What does it mean for functions to be 'eigenfunctions' and 'orthogonal'? What does it mean for a collection of functions to be a basis, and for what? Think of this as generalizing eigenvalues and orthogonal bases for $\mathbb{R}^n$. The fact that the dimension is infinite leads to serious complications.
- What are the properties of the linear operator $L[\phi] = \phi''$ that arises in the 'eigenvalue' problem $L[\phi] = \lambda \phi$?
• How can these tools be used to solve PDEs (and what are the limitations)?
• What are the implications of this theory for real problems?

1.4. Outline.
• $L^2$ and orthogonality
  ◦ functions in $L^2$ (vectors in $\mathbb{R}^n$)
  ◦ inner product, orthogonality (dot product, orthogonal vectors)
  ◦ orthogonal basis for $L^2$ (orthogonal basis for $\mathbb{R}^n$ with eigenvectors)
• Fourier series, eigenfunctions
  ◦ Appearance as solution to boundary value problem
  ◦ operator $L$ and eigenfunctions ($A$ and eigenvectors of $A$)
  ◦ Fourier series
• Heat equation, etc.
  ◦ Introduction to PDEs
  ◦ Solution using eigenfunctions

2. Inner products, orthogonality of functions

To study eigenfunctions and PDEs and so on, we need to identify the correct space where the functions of interest reside, and extend notions from linear algebra in $\mathbb{R}^n$ (see section 4) to this space of functions.

2.1. Square-integrable functions ($L^2[a, b]$). For this section, we will consider real-valued functions $f$ defined on an interval $[a, b]$. Such a function is called square-integrable$^1$ if

$$\int_a^b |f(x)|^2 \, dx < \infty.$$ 

We now define $L^2[a, b]$ to be the set of real-valued functions defined on $[a, b]$ that are square integrable, i.e.

$$L^2[a, b] = \{ f : [a, b] \to \mathbb{R} \text{ such that } \int_a^b |f(x)|^2 \, dx < \infty \}.$$ 

This space will turn out to be the right one for studying Fourier series. We will not prove it, but if $f$ and $g$ are square integrable then $f + g$ is also square integrable, so linear combinations of functions in $L^2[a, b]$ stay in $L^2[a, b]$.

**Norm:** We define a norm on the space (the $L^2$ norm) as follows:

$$\|f\| = \sqrt{\int_a^b |f(x)|^2 \, dx}.$$ 

When applied to the difference of two functions $f$ and $g$, i.e.

$$\|f - g\| = \int_a^b |f(x) - g(x)|^2 \, dx$$

the norm is a way of measuring the distance between the two functions, analogous to the Euclidean distance for vectors:

$$\|x - y\|^2 = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2.$$ 

$^1$Mathematicians will usually refer to a square integrable function as an ‘$L^2$ function’.
Notice that this ‘distance’ \( \| f - g \| \) between \( f \) and \( g \) is a sort of weighted average of the area between the curves in the interval \([a, b]\). The quantity (??) is sometimes called the mean-square distance or mean-square error if \( g \) is some approximation to \( f \).

**Norm examples:** Consider the norm for \( L^2[-1, 1] \).

If \( f(x) = x^2 \) then
\[
\| f(x) \|_2^2 = \int_{-1}^{1} x^2 \, dx = \left. \frac{1}{3} x^3 \right|_{-1}^{1} = \frac{2}{3}.
\]
If \( f(x) = |x| \) and \( g(x) = x \) then
\[
\| f(x) - g(x) \|_2^2 = \int_{-1}^{1} (x - |x|)^2 \, dx = \int_{-1}^{0} (2x)^2 \, dx = \frac{4}{3}.
\]
Note that \( f = g \) when \( x > 0 \) but they differ when \( x < 0 \).

**Notation:** For clarity, one sometimes writes \( \| f \|_2, \| f \|_{L^2} \) or \( \| f \|_{L^2[a,b]} \). Only this norm will be used, so we will just write \( \| f \| \). Be careful to keep track of the interval, because the norm depends on it; e.g. if \( f(x) = 1 \) then \( \| f \| = 2 \) in \( L^2[-1, 1] \) but \( \| f \| = 4 \) in \( L^2[-2, 2] \).

**Inner product:** The inner product on \( L^2[a, b] \) is
\[
\langle f, g \rangle = \int_{a}^{b} f(x)g(x) \, dx.
\]
Two functions are called orthogonal ‘with respect to the inner product’ if \( \langle f, g \rangle = 0 \), and a set of functions \( \{ f_n \} \) is orthogonal if distinct pairs are all orthogonal (just as with \( \mathbb{R}^n \)). Unfortunately, there is no obvious intuition for orthogonal functions (compared to \( \mathbb{R}^n \), where it means the two vectors are perpendicular).

**Notation:** Again, because this inner product is the only one we will work with, the ‘with respect to...’ part will be omitted and the inner product used will be left implied.

**Inner product examples:** Consider the norm for \( L^2[-1, 1] \). Then
\[
\langle 1, x \rangle = \int_{-1}^{1} x \, dx = 0.
\]
So the constant function 1 and the function \( x \) are orthogonal (in the interval \([-1, 1]\)).

However,
\[
\langle 1, x^2 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3},
\]
so 1 and $x^2$ are not orthogonal. On the other hand, for $g(x) = x^2 - 1/3$,
\[
\langle 1, g \rangle = \int_{-1}^{1} (x^2 - 1/3) \, dx = \frac{2}{3} - \frac{2}{3} = 0.
\]
This means that the set
\[
\{1, x, x^2 - 1/3\}
\]
is an orthogonal set in $L^2[-1,1]$, whereas
\[
\{1, x, x^2\}
\]
is not.

2.2. $L^2[-\ell, \ell]$ as a space of $2\ell$-periodic functions. A function $f(x)$ is $T$-periodic (or 'has a period $T$') if
\[
(9) \quad f(x) = f(x + T) \text{ for all } x.
\]
That is, after a length $T$, the function repeats. A periodic function is therefore defined by its values on any interval of length $T$. For example, $\sin x$ has period $2\pi$ and $\cos(4\pi x)$ has period $1/2$.

Note that the period $T$ as defined above is not unique (e.g. $\sin x$ is also periodic with period $4\pi$). The fundamental period refers to the smallest possible $T$. The distinction is not particularly important for the discussion here, however.

The point: We will often be interested in periodic functions defined for all $x \in \mathbb{R}$. If the period is $2\ell$ then we can identify such a periodic function with its restriction to $[-\ell, \ell]$. Analysis of periodic functions is then done on that interval.

Similarly, any $f \in L^2[-\ell, \ell]$ corresponds to a $2\ell$-periodic function defined by (9). That is, we construct a periodic function out of $f$ by copying it on $[\ell, 3\ell]$, then again on $[3\ell, 5\ell]$ and so on. Thus we can view $L^2[-\ell, \ell]$ also as the space of $2\ell$-periodic functions such that the integral of $f^2$ over one period is finite.

For example, consider the function $f(x) = |x|$ in $L^2[-1,1]$. This corresponds to the periodic function drawn below:

The periodic function $\sin(x)$, for instance, as a member of $L^2[-\pi, \pi]$ is continuous. If in $L^2[-\pi/2, \pi/2]$ however, $\sin x$ is not continuous (but is still in the space):
The values at the discontinuities $-\pi/2, \pi/2$ etc. are ambiguous.

3. **Fourier series: Fundamentals**

In this section we consider the space $L^2[-\ell, \ell]$ as the space of $2\ell$-periodic functions in the sense of the previous section. In terms of the theory/computations, it will not matter whether we think of $f \in L^2[-\ell, \ell]$ as really periodic on all of $\mathbb{R}$ or just defined on $[-\ell, \ell]$ but it does matter conceptually for applications.

3.1. **Definition.** Following Fourier’s example, we observe that the set of functions

\[ \text{(10)} \quad \text{constant, } \cos\left(\frac{\pi x}{\ell}\right), \cos\left(\frac{2\pi x}{\ell}\right), \cos\left(\frac{3\pi x}{\ell}\right), \ldots \quad \sin\left(\frac{\pi x}{\ell}\right), \sin\left(\frac{2\pi x}{\ell}\right), \ldots \]

forms an orthogonal set in $L^2[-\ell, \ell]$ with respect to the inner product

\[ \langle f, g \rangle = \int_{-\ell}^{\ell} f(x)g(x) \, dx. \]

Explicitly, the following identities hold (and are not too hard to prove; see homework):

\[ \text{(11)} \quad \int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \, dx = 0, \quad \text{for all } m, n, \]

\[ \text{(12)} \quad \int_{-\ell}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \text{ and } m \neq 0, \end{cases} \]

\[ \text{(13)} \quad \int_{-\ell}^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n. \end{cases} \]

Note that the orthogonality of the constant with all the sines and cosines is contained in the above (since $\cos\frac{m\pi x}{\ell} = \text{const.}$ when $m = 0$).

The **Fourier series** for a function $f \in L^2[-\ell, \ell]$ is given by

\[ \text{(14)} \quad f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right). \]

**Note:** we will study the sense in which $f$ and the series are equal later. For now, we can use the orthogonality relations (11)-(13), to formally solve for the coefficients (i.e. ignoring any possible technical issues).
For $a_m$, take the inner product of both sides with $\cos \frac{m\pi x}{\ell}$ and use the linearity property to put the inner product inside the sum:

$$\langle f, \cos \frac{m\pi x}{\ell} \rangle = \langle \frac{a_0}{2}, \cos \frac{m\pi x}{\ell} \rangle + \sum_{n=1}^{\infty} \left( a_n \langle \cos \frac{n\pi x}{\ell}, \cos \frac{m\pi x}{\ell} \rangle + b_n \langle \sin \frac{n\pi x}{\ell}, \cos \frac{m\pi x}{\ell} \rangle \right)$$

Now by (11)-(13), all the terms on the right vanish except the $n = m$ term in the cosine sum. If $n \geq 1$ then

$$\langle f, \cos \frac{m\pi x}{\ell} \rangle = a_n \langle \cos \frac{m\pi x}{\ell}, \cos \frac{m\pi x}{\ell} \rangle.$$ 

Now employ (12) to evaluate the right side. We end up with the formula

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{m\pi x}{\ell} \, dx, \quad n \geq 0. \quad (15)$$

Note that the $a_0$ case has to be checked separately. Keeping the formula valid for $n = 0$ is the reason for the choice of $1/2$ has the constant basis function. Similarly, we get

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{m\pi x}{\ell} \, dx, \quad n \geq 1. \quad (16)$$

3.2. Periodic functions and the Fourier series. Note that the basis functions $\sin\left(\frac{n\pi x}{\ell}\right)$ etc. for $L^2[-\ell, \ell]$ are periodic with period $2\ell$. Thus the Fourier series for $f(x)$ defined on $[-\ell, \ell]$ is always $2\ell$-periodic as well! This fact makes a Fourier series a natural approximation for a periodic function.

3.3. The main result. The fundamental result (which has profound implications in math and physics) is that any function $f$ in $L^2[-\ell, \ell]$ has a representation as a Fourier series.

In terms of a basis: Put another way, the set (10) of sines and cosines, $\text{constant, } \cos\left(\frac{\pi x}{\ell}\right), \cos\left(\frac{2\pi x}{\ell}\right), \cos\left(\frac{3\pi x}{\ell}\right), \cdots \sin\left(\frac{\pi x}{\ell}\right), \sin\left(\frac{2\pi x}{\ell}\right), \cdots$ forms an orthogonal basis\(^2\) for $L^2[-\ell, \ell]$, which means that

$$(17) \quad \text{Every } f \in L^2[-\ell, \ell] \text{ can be written as } f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

with $a_n, b_n$ given by (15) and (16). The equality here is not in the sense that at each point $x$, the series converges to $f(x)$; we will resolve the ambiguity later.

In terms of periodic functions: The basis functions $\sin\left(\frac{n\pi x}{\ell}\right)$ etc. for $L^2[-\ell, \ell]$ are periodic with period $2\ell$. Thus the Fourier series for $f(x)$ defined on $[-\ell, \ell]$ is always $2\ell$-periodic as well! This fact makes a Fourier series a natural approximation for a periodic function. Put yet another way, every reasonably nice periodic function is a superposition of sines and cosines.

\(^2\)Technical note just for completeness: the term ‘basis’ here is not exactly correct because it allows for infinite linear combinations; there’s a slightly different term for this.
3.4. **Partial sums; approximation and convergence (briefly).** In general, the $N$-th partial sum of the Fourier series for a periodic function $f(x)$ is

$$S_N(x) = \left( \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right)$$

This partial sum is an approximation to $f(x)$. In fact, it is often a *very good* approximation. The main result can therefore be rephrased yet again to say that every reasonably nice periodic function can be approximated by a sum of sines and cosines.

**Notation:** ‘$N$-th partial sum’ might instead be defined to be the first $N$ *non-zero terms* of the series rather than terms up to $\cos N\pi x/\ell$ and $\sin N\pi x/\ell$; it depends on the series in question.

This notion of approximation lets us describe the equality in precisely; it means that

$$\|f - S_N\| \to 0 \text{ as } N \to \infty.$$ 

That is, the ‘mean square error’ (i.e. the distance between $f$ and $S_N$ as measured in the $L^2$ norm)

$$\|f - S_N\|^2 = \int_{-\ell}^{\ell} |f(x) - S_N(x)|^2 \, dx$$

goes to zero as we increase the number of terms in the approximation.

Because the error takes into account the difference $|f(x) - S_N(x)|$ at every point in the interval, we would like to say that the convergence implies that the error goes to zero uniformly, i.e.

$$\max_{x \in [-\ell, \ell]} |f(x) - S_N(x)| \to 0.$$

One might care about the above if trying to use a Fourier series to approximate a function (it should be a good estimate everywhere!). If $f(x)$ is nice enough this is true, but it is *not true in general*. The distinction between the various notions of convergence is subtle; we will consider it in more detail later after becoming better acquainted with Fourier series.

3.5. **Examples.** Some examples help to illustrate the definitions of the previous section.

3.5.1. **Triangular wave.** Let

$$f(x) = \begin{cases} 
-x & -1 \leq x < 0 \\
x & 0 < x \leq 1 
\end{cases}$$

and $f(x) = f(x + 2)$ when $x \notin [-1, 1]$. This is the same as before:
To compute the Fourier series (note that $\ell = 1$), we use (11)-(13). Some integration by parts is involved; in particular the formula
\[
\int x \cos ax \, dx = \frac{1}{a} x \sin ax + \frac{1}{a^2} \cos ax.
\]
When $n \geq 1$,
\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos(n\pi x) \, dx
\]
\[
= -\int_{-1}^{0} x \cos(n\pi x) \, dx + \int_{0}^{1} x \cos(n\pi x) \, dx
\]
(change $x \to -x$ in first term)
\[
= 2 \int_{0}^{1} x \cos(n\pi x) \, dx
\]
\[
= \left[ \frac{2}{n\pi} x \sin(n\pi x) + \frac{2}{n^2\pi^2} \cos(n\pi x) \right]_{0}^{1}
\]
(since $\sin(n\pi) = 0$ for all $n$)
\[
= \frac{2}{n^2\pi^2} \cos(n\pi x) \bigg|_{0}^{1}
\]
\[
= \frac{2}{n^2\pi^2} ((-1)^n - 1).
\]
Thus for $n \geq 1$,
\[
a_n = \begin{cases} 
-\frac{4}{n^2\pi^2} & \text{for odd } n \\
0 & \text{for even } n
\end{cases}
\]
For $a_0$ just compute
\[
a_0 = \int_{-1}^{1} f(x) \, dx = 2 \int_{0}^{1} x \, dx = 1.
\]
For the sine terms, there is cancellation:
\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin(n\pi x) \, dx
\]
\[
= -\int_{-1}^{0} x \sin(n\pi x) \, dx + \int_{0}^{1} x \sin(n\pi x) \, dx
\]
\[
= 0.
\]
The two terms exactly cancel. The Fourier series representation for $f$ is therefore
\[
f(x) = \frac{1}{2} \bigg( -\frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\pi x) \bigg).
\]
Now let’s define the partial sum containing $N$ non-zero terms as follows (not the same as the general definition earlier to avoid terms that are zero):
\[
S_N(x) = \frac{1}{2} \bigg( -\frac{4}{\pi^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2} \cos((2n-1)\pi x) \bigg)
\]
so $S_1 = \frac{1}{2} - \frac{4}{\pi^2} \cos \pi x, S_2 = \frac{1}{2} - \frac{4}{\pi^2} (\cos \pi x + \frac{1}{9} \cos 3\pi x)$ and so on.
A plot of the approximations shows that the agreement is quite good, even with only a few terms:

\[
\| f - S_N \|^2 = \int_{-1}^{1} |f(x) - S_N(x)|^2 \, dx
\]

shows that the mean-square error does indeed go to zero as \( N \to \infty \). In fact, it goes to zero rather quickly (exponentially fast!). The maximum error

\[
\max_{x \in [-\ell, \ell]} |f(x) - S_N(x)|
\]

also goes to zero as \( N \to \infty \), but much slower; the agreement is not so good near the peak of the triangle where there is a sharp corner.
3.5.2. Square wave. Let

\[ f(x) = \begin{cases} 
-1 & -1 \leq x < 0 \\
1 & 0 < x \leq 1 
\end{cases} \]

and \( f(x) = f(x + 2) \) when \( x \notin [-1, 1] \). Then \( a_0 = 0 \) and

\[
a_n = \int_{-1}^{1} f(x) \cos(n\pi x) \, dx = -\int_{-1}^{0} \cos(n\pi x) \, dx + \int_{0}^{1} \cos(n\pi x) \, dx = 0,
\]

\[
b_n = \int_{-1}^{1} f(x) \sin(n\pi x) \, dx \\
= \int_{-1}^{0} \sin(n\pi x) \, dx + \int_{0}^{1} \sin(n\pi x) \, dx \\
= 2 \int_{0}^{1} \sin(n\pi x) \, dx \\
= -\frac{2}{n\pi} \cos(n\pi x) \bigg|_{0}^{1} \\
= -\frac{2}{n\pi}((-1)^n - 1)
\]

so \( b_n = \frac{4}{n\pi} \) when \( n \) is odd and \( b_n = 0 \) when \( n \) is even. Thus the Fourier series for \( f(x) \) is

\[ f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi x). \]

The \( N \)th partial sum \( S_N \) is

\[ S_N(x) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{2n-1} \sin((2n-1)\pi x). \]

Error: A plot of the approximation shows that the partial sums converge nicely where \( f \) is continuous, but do not perform well at all near the discontinuity:
The partial sums tend to oscillate and overshoot the discontinuity by a significant amount. Again, define $S_m(x)$ to be the first $m$ terms of the series above, e.g. $S_2 = \frac{4}{\pi}(\sin \pi x + \frac{1}{3} \sin 3\pi x)$. Then as in the previous example,
\[
\|f - S_N\| \to 0 \text{ as } N \to \infty.
\]
Zooming in on the top part of the discontinuity at $x = 1$:

As the number of terms increases, the overshoot does not decrease in magnitude. The amplitude of the bad oscillations stays about the same; the width, however, shrinks. Thus at each point $x \neq 1$, the series will converge to $f(x)$, but the maximum error is
\[
\max_{x \in [-1,1]} |f(x) - S_N(x)| \approx 0.18 \text{ as } N \to \infty.
\]
4. Review: Inner products and orthogonality in \( \mathbb{R}^n \)

Reviewing the relevant ideas in \( \mathbb{R}^n \) is helpful. Recall that the 2-norm (or Euclidean norm) of a vector \( \mathbf{x} \) is
\[
\| \mathbf{x} \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\]
The quantity \( \| \mathbf{x} - \mathbf{y} \| \) measures the distance between two vectors in \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) (which is literally the distance between the two points \( x \) and \( y \)). The inner product (or dot product) of two vectors is
\[
\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n
\]
also denoted \( \mathbf{x} \cdot \mathbf{y} \). The inner product on \( \mathbb{R}^n \) has several important properties:

i) **Symmetry:**
\[
\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle.
\]

ii) **Linearity in each argument:** If \( \mathbf{x}, \mathbf{y}, \mathbf{z} \) are vectors then
\[
\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \quad \text{and} \quad \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle.
\]
If \( c \) is a scalar then
\[
\langle c \mathbf{x}, \mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle \quad \text{and} \quad \langle \mathbf{x}, c \mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle.
\]

iii) **Positive definiteness:**
\[
\langle \mathbf{x}, \mathbf{x} \rangle \geq 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{x} \rangle = 0 \implies \mathbf{x} = 0.
\]

Note that the Euclidean norm is given by
\[
\| \mathbf{x} \| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.
\]
Property (iii) ensures that \( \| \mathbf{x} \| = 0 \) if and only if \( \mathbf{x} = 0 \).

Two vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) are **orthogonal** if their inner product is zero:
\[
\mathbf{x}, \mathbf{y} \text{ orthogonal} \iff \langle \mathbf{x}, \mathbf{y} \rangle = 0.
\]
A basis \( \mathbf{v}_1, \cdots, \mathbf{v}_n \) for \( \mathbb{R}^n \) is called **orthogonal** if every pair of distinct basis elements is orthogonal, i.e.
\[
\{ \mathbf{v}_j \} \text{ is an orthogonal basis if } \langle \mathbf{v}_j, \mathbf{v}_k \rangle = 0 \text{ for } j \neq k.
\]
5. A Preview of Future Topics

In the coming weeks, we will develop a method inspired by the above to solve

\[ u_t = -L[u] + h, \quad x \in (a, b) \text{ and } t > 0 \]

with some initial condition \( u(x, 0) = u_0(x) \) and boundary conditions at \( x = a \) and \( x = b \).

The differential operator \( L \) contains only \( x \)-derivatives, e.g. \( L = -\frac{d^2}{dx^2} \). It will be the same sort of operator we encountered in studying second-order linear equations.

We can identify some key step from Fourier’s heat conduction problem:

A) Find ‘eigenfunctions’ that solve a boundary value problem of the form

\[ L[\phi_n] = \lambda_n \phi_n, \quad \text{with some conditions on } \phi \text{ at } x = a \text{ and } x = b. \]

B) Represent the solution as a sum of coefficients (depending on time) times eigenfunctions:

\[ u = \sum_{n=0}^{\infty} a_n(t) \phi_n(x). \]

C) Find (simple) equations for the coefficients and solve them to get the solution.

To make (C) work, we will need the functions \( \phi_n \) to have specific properties, namely, something like (7) that will make sure the equation for \( c_n(t) \) decouples from the others. In Fourier’s example, the equation is

\[ a'_n = -\lambda_n a_n, \quad a_n(0) = \text{coeff. of } \phi_n \text{ in the initial condition} \]

and the ‘eigenvalue’ is \( \lambda_n = n^2 \).

In a sense, we are converting (complicated) linear PDEs into a linear algebra problem plus an ODE problem. Doing so will require developing substantial theory in several areas:

- **Vector spaces of functions, orthogonality:** How do we make sense of series like

\[ \sum_{n=0}^{\infty} c_n \phi_n(x) \]

and extend the notion of a basis to functions? Where does the condition (7) come from? Essentially, we will extend familiar results from linear algebra in \( \mathbb{R}^n \) to spaces of functions.

- **Boundary value problems / eigenvalue problems:** What is the structure of ODEs when the boundary conditions are at different points (not IVPs)? These problems are quite different from the IVPs of previous weeks! We will also develop theory for eigenvalue problems (what properties do the eigenvalues and eigenfunctions have?).

- **Important PDEs (what important equations can be solved this way?):** There are three essential PDEs (the heat equation, wave equation, and Laplace’s equation) that can be studied using the framework we develop here. These three equations describe a wide range of phenomena in physics, engineering, and more.
• **Fourier series (an extremely important special case):** The case where the eigenfunctions are sines/cosines is of critical importance - with implications that go far beyond their use as a tool for solving PDEs. We will look at such series (Fourier series) in detail.