MATH 353 MIDTERM II: ADDITIONAL REVIEW PROBLEMS

MIDTERM DATE: THURSDAY, NOV. 15

Notes
The problems here are a mix of computational and conceptual problems. They are intended to give you a rough sense of the sort of problems you will see on the exam. The suggested book problems in the Midterm guide and the pre-midterm homework cover most of the material, so I have provided a limited set of additional problems here. This set of problems does not cover everything on the exam.

Solutions will be posted at a later date.

Problems (not PDEs)

R1. Consider the sawtooth wave defined by

\[ f(t) = t, \quad t \in [0, 1) \quad \text{and} \quad f(t + 1) = f(t) \] for \( t \geq 0 \).

a) Draw \( f(t) \).

b) Write \( f(t) \) as an infinite sum involving step functions.

c) Compute the Laplace transform of \( f \) (you may leave this as an infinite series).

R2. Let

\[ f(t) = \begin{cases} \sin t/t & t \neq 0 \\ 0 & t = 0 \end{cases} . \]

a) Find the power series for \( f(t) \) around \( t = 0 \).

b) By transforming term by term, find the Laplace transform of \( f(t) \) in the form of an infinite sum. You may assume this is valid, everything converges etc.

c) (added) Use the rule for \( F'(s) \) to compute the Laplace transform of \( f(t) \). To get the integration constant, you will need to show that

\[ \lim_{s \to \infty} F(s) = 0 \]

which can be done using (b).
R3. Determine the recurrence relation for the power series solution to
\[ y'' + (x^2 + 1)y' + y = 0 \]
at \( x_0 = 0 \). Find the terms of up to \( x^2 \) of two linearly independent solutions \( y_1 \) and \( y_2 \).

R4. Using the Laplace transform, solve the initial value problem
\[ y'' - 4y = \delta(t - 1) + u(t - 1), \quad y(0) = y'(0) = 0 \]
where \( u \) is the unit step function.

R5. a) Suppose \( Y(s) \) is the Laplace transform of the solution \( y(t) \) to a linear, constant coefficient ODE and
\[ Y(s) = \frac{1 + 2s}{(s - r_1)(s - r_2)(s - r_3)} \]
where \( r_1, r_2, r_3 \) are distinct real numbers. Under what conditions is \( \lim_{t \to \infty} y(t) = 0 \)?
b) Find an IVP for a constant-coefficient, linear ODE whose solution \( y(t) \) has Laplace transform
\[ Y(s) = \frac{1 - 2s + s^2}{s^3 - 1} + \frac{e^{-2s}}{s^3}. \]

R6 (short questions). Answer the following questions.

a) Solve the ODE
\[ x'(t) = \delta(t), \quad x(0) = 0. \]

b) Find the radius of convergence of the series
\[ \sum_{n=0}^{\infty} \frac{n!}{n^n} x^{2n}. \]

*Hint:* you may need the limit \( \lim_{n \to \infty} (1 + 1/n)^n = e \).

c) Find the power series at \( x_0 = 0 \) and the radius of convergence for \( f(x) = (x + 1)^2 \).

d) Suppose \( f \) satisfies, for all \( t > 0 \),
\[ \int_0^t sf(t - s) \, ds = t. \]
Does such an \( f \) exist? If so, find it.

Problems (PDEs)

Q1. Solve the initial boundary value problem
\[ u_t = 4u_{xx}, \quad x \in (0, 1), \ t > 0 \]
\[ u_x(0, t) = u_x(1, t) = 0, \quad t > 0 \]
\[ u(x, 0) = 3 \cos(\pi x) - 2 \cos(4\pi x). \]
You may assume there are no negative eigenvalues.
Q2. Consider the initial boundary value problem

\[ u_t = 3t^2 u_{xx}, \quad x \in (0, 1/2), \quad t > 0 \]
\[ u(0, t) = u_x(1/2, t) = 0, \quad t > 0 \]
\[ u(x, 0) = x. \]

Find the solution. You may assume the eigenvalues are positive. Leave constants \((a_n, b_n\text{ etc.})\) as explicit integrals that you could evaluate directly (but do not compute them), e.g. expressions like

\[ a_n = \int_0^{1/2} x^2 \cos(nx) \, dx. \]

Q3. Consider the initial boundary value problem

\[ u_t = u_{xx}, \quad x \in (0, 2), \quad t > 0 \]
\[ u_x(0, t) = u_x(2, t) = 0, \quad t > 0 \]
\[ u(x, 0) = \sin(\pi x/2). \]

a) Find the solution. You may assume there are no negative eigenvalues. Leave your solution in the form

\[ u(x, t) = \sum_{n=0}^{\infty} b_n e^{-\lambda_n t} \phi_n(x), \]

giving explicit formulas for \(b_n, \lambda_n\) and \(\phi_n\). Write down equations for the \(b_n\)'s but do not evaluate the integrals.

b) Using (a), show that the solution converges to a uniform value as \(t \to \infty\), i.e.

\[ \bar{u} = \lim_{t \to \infty} u(x, t). \]

for a constant \(\bar{u}\). Compute its value.

Q4 (fundamentals).

a) Show that the solution to the IBVP

\[ u_t = u_{xx}, \quad x \in (0, 1), \quad t > 0 \]
\[ u(0, t) = 1, \quad u(1, t) = t^2, \quad t > 0 \]
\[ u(x, 0) = x \]

can be written as the sum of two solutions (to the same PDE), one of which has homogeneous boundary conditions and the other of which has zero initial conditions.
b) For each of the following PDEs, identify whether it is linear. If so, also identify whether it is homogeneous or inhomogeneous.

i) \( u_{tt} = u_{xx} + \sin t \)
ii) \( u_t + uu_x = u_{xx} \)
iii) \( x\psi_{xx} + y\psi_{yy} = 0 \)
iv) \( u_t = u_{xx} + u(1 - u) \)

Here \( u(x,t) \) is a function of \( x,t \) and \( \psi(x,y) \) is a function of \( x \) and \( y \).

**Q5 (true/false).** Identify whether the following statements are true or false. If false, briefly explain why.

a) The functions 1, \( x \) and \( x^2 \) are orthogonal in \( L^2[-1,1] \).

b) If \( f \in L^2[a,b] \) then \( \langle f,f \rangle \geq 0 \).

c) Every solution to the IBVP

\[
\begin{align*}
    u_t &= u_{xx}, \quad x \in (0, \pi), \ t > 0 \\
    u_x(0,t) &= 0, \quad u_x(\pi,t) = 0, \quad t > 0 \\
    u(x,0) &= f(x)
\end{align*}
\]

converges to zero as \( t \to \infty \).

d) The boundary value problem

\[
y'' + y = 0, \quad y(0) = a, \quad y(1) = b
\]

must have either no solutions, one solution or infinitely many solutions.

**Q6.** Show that the eigenvalue problem

\[
-\phi'' = \lambda \phi, \quad \phi(0) = 0, \quad \lim_{x \to \infty} \phi(x) = 0
\]

for functions on \([0, \infty)\) has no solutions (for any \( \lambda \)).

**Q7.** Suppose \( \phi_1, \phi_2, \ldots, \phi_n \) are orthogonal functions in \( L^2[0,1] \) and

\[
f = c_1 \phi_1 + c_2 \phi_2 + \cdots + c_n \phi_n.
\]

Find a simple formula for \( \langle f, f \rangle \) in terms of the coefficients.