Note: It is good practice to do all the calculations (partial fractions, convolution integrals) by hand. Variants of such calculations may show up on the exam (but you will, of course, have access to a table of standard integrals and the transforms in the textbook table).

Problems:

- Complete the review problem for Thursday (not turned in, but important; the lecture will be hard to follow otherwise). As with all the review problems, the questions are just there to indicate the sorts of things you are expected to know.

- Section 6.5: 9 13, Note: plots do not have to be done by hand.

- Section 6.6: 1ab, 9, 13

- AHP 58. Note: Write the integral as a convolution. You will need to factor out $e^{-t}$ from the integral to get $e^{-t}(...)$ and use more rules to deal with the $e^{-t}$.

Additional problems:

Review: Power series [ungraded]. You should review power series (Taylor series) for functions:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(0).$$

a) What does it mean for a power series to converge at a point $x$?

b) Find the power series for $e^{-x^2}$ and $\sin x/x$. (Note: find the series for $e^x$ for $\sin x$ first, then plug in $-x^2$ for the former and divide by $x$ for the latter).

c) By adding the series for $e^x$ and $e^{-x}$, find the power series for

$$\cosh x = \frac{1}{2} (e^x + e^{-x}).$$

Note: you could just differentiate, but the point is to do so by manipulating other series.

d) Use the ratio test to show that the series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

converges for all $x$. 

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e) Use the ratio test to show that

\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} x^n \]

converges for \(|x| < 1\).

**P1 (resonance).** Recall that we showed that resonance (oscillations that grow in amplitude over time) occurs when an oscillator is forced in sync with its natural frequency. Here you will see if the same occurs when instantaneous pushes are used instead of sines.

Consider the initial value problem

\[ y'' + y = \sin \omega t, \quad y(0) = y'(0) = 0. \]

The Laplace transform of the solution (from class) is

\[ Y(s) = \frac{\omega}{(s^2 + 1)(s^2 + \omega^2)}. \]

a) Use the inverse transform to compute \(y(t)\) when \(\omega \neq 1\).

b) Do the same as (a) for \(\omega = 1\). \(^1\) Is there resonance?

c) Find the Laplace transform of the ‘impulse train’

\[ g_{\text{tr}}(t) = \delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \cdots = \sum_{n=0}^{\infty} \delta(t - 2\pi n) \]

in the form of an infinite sum and as a single expression.

d) Find the solution \(y(t)\) to

\[ y'' + y = g_{\text{tr}}(t), \quad y(0) = y'(0) = 0. \]

Note: you should use the infinite sum form of (c) and inverse transform term by term.

e) Show that \(y(t)\) is unbounded as \(t \to \infty\). **Hint: write down an expression for \(y(t)\) after \(N\) impulses have been applied as we did in class for \(N = 2\).**

\(^1\)Note: the result is in the lecture notes / you can use a computer if you don’t want to compute it. If computing by hand, use the convolution theorem.
SECOND part

Book problems:

• Section 5.1: 5, 25

P2 (series solution). Here we consider power series solutions to the IVP

\[ y'' - y = x^3, \quad y(0) = 1, \quad y'(0) = 1. \]

a) Look for a power series solution around \( x_0 = 0 \). Note: use the initial conditions to determine \( a_0 \) and \( a_1 \); be careful with the \( x^3 \) term. Obtain the recurrence relation and find the solution, up to and including the \( x^5 \) term (you do not need to find the terms past this).

b) Plot the series solution (keeping only the terms up to \( x^5 \)) and the exact solution

\[ y(x) = -x(x^2 + 6) - 3e^{-x} + 4e^x. \]

On what interval does it appear to be a reasonable approximation?