Topics covered

- Impulse
  - Dirac delta properties
  - ODEs with impulse forcing
  - Impulse response
- Inverse transform examples
  - Suggestions for computations
  - Some miscellaneous rules

1 Dirac delta and impulse, continued

1.1 The Dirac delta (from last week)

We define an object $\delta(t)$, called the Dirac delta that obeys the following rules:

\[(i) \quad \int_{-\infty}^{\infty} \delta(t) \, dt = 1,\]

\[(ii) \quad \delta(t) = 0 \quad \text{for } t \neq 0,\]

\[(iii) \quad \int_{-\infty}^{\infty} \delta(t-a)f(t) \, dt = f(a) \quad \text{for all continuous functions } f(t).\]

It follows from (iii) that $\delta$ is the analogue of the multiplicative identity for convolutions:

$$\delta * f = f * \delta = f.$$
If \(c > 0\) then
\[
\mathcal{L}[\delta(t - c)] = \int_0^\infty e^{-st} \delta(t - c) \, dt = e^{-cs},
\]
and
\[
\mathcal{L}[\delta] = 1.
\]
(See last week’s notes for more details).

### 1.2 Other properties of \(\delta\)

An impulse of size \(a\) at time \(c\) would be given by
\[
a\delta(t - c)
\]
i.e. it makes sense to multiply \(\delta\) by a constant factor. This just changes the total impulse contained in the delta expression:
\[
\int_{-\infty}^{\infty} a\delta(t - c) \, dt = a.
\]
However, it is not obvious how to multiply \(\delta\) by a function. The rule, equivalent to property (iii), is that because \(\delta(t) = 0\) when \(t \neq 0\),
\[
f(t)\delta(t - a) = f(a)\delta(t - a).
\]
In reality, \(f(t)\delta(t - a)\) does not make rigorous sense; the above really means that
\[
\int_{-\infty}^{\infty} (f(t) - f(a))\delta(t - a) \, dt = 0.
\]
The ‘identity’ (1) may be a useful way to remember property (iii); assuming \(\delta\) is a function and using (1) will give you the right answers:
\[
\int_{-\infty}^{\infty} f(t)\delta(t - a) \, dt = \int_{-\infty}^{\infty} f(a)\delta(t - a) \, dt = f(a) \int_{-\infty}^{\infty} \delta(t - a) \, dt = f(a).
\]

(Optional: Verifying a \(\delta\)-property with a limit) Recall that all of the \(\delta\) identities in an integral can be viewed as shorthand for replacing \(\delta\) with \(g_n(t)\) (the approximations) and then taking \(n \to \infty\), where
\[
g_n(t) = \begin{cases} 
  n/2 & -1/n < t < 1/n \\
  0 & \text{otherwise}
\end{cases}
\]
The functions \(g_n(t)\) are boxes centered around zero that shrink in width as \(n \to \infty\) but have \(\int_{-\infty}^{\infty} g_n(t) \, dt = 1\). We can verify property (iii) using the interpretation of \(\delta\) as a limit of approximations \(g_n(t)\), in the sense that
\[
\int \delta(\cdots) \, dt \text{ means } \lim_{n \to \infty} \int g_n(\cdots) \, dt.
\]
In this sense, property (iii) reads
\[ \int_{-\infty}^{\infty} \delta(t-a)f(t) dt = \lim_{n \to \infty} g_n(t-a)f(t) dt = \lim_{n \to \infty} \frac{1}{2/n} \int_{-1/n}^{1/n} f(t) dt. \]

This rightmost limit is the average value of \( f \) on \([a - 1/n, a + 1/n]\) as \( n \to \infty \). Since \( f \) is continuous, as \( n \to \infty \) its value on the interval goes to \( f(a) \), so the average also goes to \( f(a) \). Rigorously, we would use the mean value theorem for integrals.

### 1.3 Transform of \( \delta \); instantaneous forcing

As noted in the last section, \( \mathcal{L}[\delta(t-c)] = e^{-cs} \), \( c > 0 \)
and \( \mathcal{L}[\delta(t)] = 1 \). This allows us to solve IVPs with an instantaneous forcing. The process is the same as in previous sections; we just need to use the \( \mathcal{L}[f(t-c)u_c(t)] = e^{-cs}F(s) \) rule when taking the inverse transforms to deal with the negative exponentials introduced by the \( \delta \)'s.

Notably, even though we only have formal rules for manipulating \( \delta \), we end up with the right answer; moreover, \( y(t) \) will contain only actual functions and no \( \delta \)'s so there will be no ambiguity in its value. As an example, consider
\[ y'' + y' + 2y = \delta(t-1), \quad y(0) = 1, y'(0) = 0. \]

Applying the Laplace transform, we find that
\[ (s^2 + s + 2)Y - s = e^{-s}, \]
and so
\[ Y(s) = \frac{s}{(s-1)(s+2)} + e^{-s}H(s), \quad H(s) = \frac{1}{(s-1)(s+2)}. \]

The first term is the homogeneous part; the second is the response to the \( \delta \) forcing. We then apply the rule \( \mathcal{L}[f(t-c)u_c(t)] = e^{-cs}F(s) \) to obtain the inverse transform:
\[ y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t} + h(t-1)u_1(t) \]
where \( h = \mathcal{L}^{-1}[H] = \frac{1}{3}e^{-t} - \frac{1}{3}e^{2t} \), which is obtained from partial fractions:
\[ H(s) = \frac{1/3}{s - 1} - \frac{1/3}{s + 2}. \]

The following example is somewhat less arbitrary:
Getting a pendulum to swing: A pendulum with angular displacement \( y(t) \) is governed (approximately) by the ODE

\[ y'' + y = g(t) \]

where \( g \) is the applied force. If it is let go at \( y(0) = 1 \) then the pendulum will swing back and forth from \(-1\) to \(1\) with period \(2\pi\). Suppose we push the pendulum after it swings back to \( y(0) = 1 \) at \( t = 2\pi \) with an impulse of 1, then again at \( t = 4\pi \) with the same impulse. Then \( y(t) \) is the solution to

\[ y'' + y = \delta(t - \pi) + \delta(t - 2\pi), \quad y(0) = 1, y'(0) = 0. \]

Take the Laplace transform of the IVP:

\[ s^2 Y + s = e^{-2\pi s} + e^{-4\pi s}. \]

This gives

\[ Y = -\frac{s}{s^2 + 1} + H(s)(e^{-2\pi s} + e^{-4\pi s}), \quad H(s) = \frac{1}{s^2 + 1}. \]

The inverse transform of the first term is just \(-\cos t\), which is the homogeneous solution. For the second two terms, we use the step function transform rule:

\[ \mathcal{L}[f(t-c)u_c(t)] = e^{-cs}F(s) \]

to obtain

\[ y(t) = -\cos t + h(t-2\pi)u_{2\pi}(t) + h(t-4\pi)u_{4\pi}(t). \]

The inverse transform of \( H \) is \( h(t) = \mathcal{L}^{-1}[H(s)] = \sin t \) and \( \sin(t-2\pi) = \sin t \) so

\[ y(t) = -\cos t + \sin(t)(u_{2\pi}(t) + u_{4\pi}(t)). \]

Observe that

\[ y(t) = -\cos t + \begin{cases} 0 & t < 2\pi \\ \sin t & 2\pi < t < 4\pi \\ 2\sin t & t > 4\pi \end{cases}. \]

Notice that after the first push, there is a new oscillation; then after the second, this response has twice the amplitude. If the process continues, the amplitude will continue to grow (see homework).
1.4 (optional:) Impulse response

Recall that the IVP

\[ ay'' + by' + cy = g(t), \quad y(0) = y'(0) = 0 \]

has a transformed solution

\[ Y(s) = H(s)G(s) \]

where \( H(s) = 1/(as^2 + bs + c) \) is the transfer function. We used this and the convolution theorem to then show that the solution to the IVP is

\[ y(t) = (h \ast g)(t) \]

where \( h(t) = \mathcal{L}^{-1}[H(s)]. \)

Now observe that when \( g = \delta(t), \) the transform/solution are simply

\[ Y(s) = H(s), \quad y(t) = h(t). \]

For this reason, the function \( h(t) \) is called the impulse response of the system: the response to a unit impulse with zero initial conditions.

Restating the result for the solution for any \( g, \) we have now shown an important principle:

For a system governed by a linear constant coefficient ODE, the response of the system to an input \( g(t) \) is the convolution of the impulse response with the input.

Remark: In fact, this idea - of constructing a solution \( h \) to a equation with \( \delta \) forcing, then taking a convolution \( h \ast g \) to get a solution for any forcing \( g \) - is a powerful tool for solving linear DEs (even without the Laplace transform). In the context of this theory, \( h(t) \) is called a Green’s function.
2 Computing the inverse transform

With the basic transforms established, let’s look at some examples of calculating the inverse, making use of the various tools developed so far. (Note: in reading the examples, you may want to keep the table of standard transforms open). In many cases, we proceed by recognizing which is the ‘base’ part that needs to be evaluated directly (usually via the table), and which factors correspond to modifying the base part (differentiation, translation etc.).

A few essential transforms to know are

\[\mathcal{L}[e^{at}] = \frac{1}{s - a}, \quad \mathcal{L}[1] = \frac{1}{s},\]
\[\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \quad \mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}.\]

Note that since the \(e^{at}\) formula works for complex \(a\), the sine/cosine transforms are really just the real/imaginary parts of \(\mathcal{L}[e^{iat}]\). That is,

\[\mathcal{L}[e^{iat}] = \frac{1}{s - ia} = \frac{s + ia}{s^2 + a^2} = \frac{s}{s^2 + a^2} + i \left(\frac{a}{s^2 + a^2}\right).\]

Thus when doing transforms/inverse transforms it is okay to keep things in complex form and then take real parts at the end, e.g.

\[\mathcal{L}^{-1} \left[\frac{s}{s^2 + 1}\right] = \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s - i} + \frac{1}{s + i}\right] = \frac{1}{2} (e^{it} + e^{-it}) = \cos t.\]

A selected summary of rules is as follows:

\[\mathcal{L}[f'(t)] = sF(s) - f(0)\] (2)
\[\mathcal{L}[(-t)^nf(t)] = F^{(n)}(s)\] (3)
\[\mathcal{L}[f(t - c)u_c(t)] = e^{-cs}F(s)\] (4)
\[\mathcal{L}[e^{ct}f(t)] = F(s - c)\] (5)
\[\mathcal{L}[f(at)] = \frac{1}{a} F(s/a)\] (6)

along with the rules from major theorems of the section (convolutions, \(\delta\), higher-order derivatives etc.).

2.1 Examples using the transform rules

Example 1: We use some rules to find the transform of

\[\mathcal{L}[te^{2t} \sin t].\]

Observe that multiplying by \(t = -(\mathcal{L}^{-1})\) corresponds to \(-\frac{d}{ds}\) and \(e^{2t}\) to a translation right by 2. Thus we need only compute \(\mathcal{L}[\sin t]\), then translate it by 2 (the \(e^{2t}\) part), then apply \(-\frac{d}{ds}\).

Let \(\mathcal{L}[\sin t] = F(s) = 1/(s^2 + 1)\). Then

\[\mathcal{L}[te^{2t} \sin t] = -F'(s - 2) = -\frac{d}{ds} \frac{1}{(s - 2)^2 + 1} = \frac{2(s - 2)}{((s - 2)^2 + 1)^2}.\]
Of course, if given the transform for $e^{at} \sin bt$ this is even easier.

**Example 2:** To compute $L^{-1}$ of

$$F(s) = \frac{e^{-s}}{s^2 - 1},$$

we recognize that the $e^{-s}$ corresponds to a translation by 1 (times the step function), so

$$L^{-1}[F(s)] = f(t-1)u_1(t)$$

where $f(t)$ is the inverse transform of $1/(s^2 - 1)$. Via partial fractions,

$$L^{-1}\left[ \frac{1}{s^2 - 1} \right] = L^{-1}\left[ \frac{1/2}{s - 1} \right] - L^{-1}\left[ \frac{1/2}{s + 1} \right] = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t.$$ 

It follows that

$$L^{-1}[F(s)] = \sinh(t-1)u_1(t).$$

**Example 3:** To compute $L[t^n]$ we can use the $(-t) \iff d/ds$ rule. Let $F(s) = L[1] = 1/s$.

Then $t^n$ is $n$ factors of $(-t)$ with an extra $(-1)^n$, so

$$L[t^n] = (-1)^n L[(-t)^n \cdot 1] = (-1)^n F^{(n)}(s) = \frac{n!}{s^{n+1}}.$$ 

### 2.2 Partial fractions

Here we inverse transform the function

$$F(s) = \frac{P(s)}{Q(s)}$$

where $Q$ has larger degree than $P$. Note: if $P, Q$ have the same degree then subtract a constant, e.g.

$$\frac{s^2}{s^2 - 1} = 1 + \frac{1}{s^2 - 1}.$$ 

If $P$ has a larger degree than $L^{-1}[F]$ will involve $\delta'$ (this will not come up).

**Simple roots:** If the denominator is the product of linear factors,

$$F(s) = \frac{P(s)}{(s - r_1)(s - r_2) \cdots (s - r_n)}$$

then applying partial fractions is straightforward:

$$\frac{P(s)}{(s - r_1)(s - r_2) \cdots (s - r_n)} = \frac{a_1}{s - r_1} + \cdots + \frac{a_n}{s - r_n}.$$ 

Multiply both sides by the denominator:

$$P(s) = a_1(s - r_2) \cdots (s - r_n) + \cdots + a_n(s - r_1) \cdots (s - r_{n-1}).$$
Now equate the LHS and RHS as polynomials and solve for the coefficients. One trick: Often you can find some or all of the $a_i$’s by testing at $s = r_i$ (most of the terms will vanish). For instance:
\[
\frac{s}{(s-1)(s+2)} = \frac{a}{s-1} + \frac{b}{s+2} \implies s = a(s+2) + b(s-1).
\]
Either equate coefficients at each power of $s$ to find $a + b = 1$ and $2a - b = 0$ or substitute in $s = -2$ to get $-2 = b(-2 - 1)$ and $s = 1$ to obtain $1 = a(1 + 2)$ (so $a = 1/3$ and $b = 2/3$).

**Repeated roots:** If $r$ is repeated $k$ times then we need one term for each power of $(s - r)$ up to $k$:
\[
F(s) = \frac{\cdots}{(s-r_1)^k \cdots} = \frac{a_1}{s - r_1} + \frac{a_2}{(s-r_1)^2} + \cdots + \frac{a_k}{(s-r_1)^k} + \cdots.
\]
To inverse transform, we need to use the formula
\[
\mathcal{L}^{-1}[1/(s-r_1)^j] = e^{rt} \mathcal{L}^{-1}[1/s^j] = \frac{1}{(j-1)!} t^{j-1} e^{rt}.
\]
The above is derived using the translation in $s$ rule ($e^{ct} f(t) \implies \mathcal{F}(s-c)$) and the standard transform for $1/s^j$. It is also in Table 6.2.1. For example, consider
\[
F(s) = \frac{1}{s^2 + 2s + 2}.
\]
Expand $s^2$ around $s = 1$ to give
\[
F(s) = \frac{(s-1)^2 + 2(s-1) + 1}{(s-1)^3} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3}.
\]
The inverse transform is then
\[
\mathcal{L}^{-1}[F] = e^t + 2te^t + \frac{1}{2} t^2 e^t.
\]

**Complex roots:** One can either factor the complex roots too and end up with only linear factors (use the above cases, but $a_1, a_2, \cdots$ are now complex), or leave them as quadratics. **Note that for a quadratic in the denominator,** you need $a + bs$ in the numerator (two unknowns). For instance, consider
\[
F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{b + cs}{s^2 + 2s + 2}.
\]
To inverse transform, we need the second term to include
\[
\frac{1}{(s-c)^2 + 1} = \mathcal{L}[e^{-ct} \sin t] \quad \text{or} \quad \frac{s-c}{(s-c)^2 + 1} = \mathcal{L}[e^{-ct} \cos t].
\]
This is done by completing the square, so we really should apply partial fractions like so:
\[
F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{b + c(s+1)}{(s+1)^2 + 1}.
\]
Solving for the coefficients yields \( a = 1/2, b = c = -1/2 \). An alternate approach would be to use complex numbers in the partial fractions expansion:

\[
F(s) = \frac{a}{s} + \frac{b}{s - (-1 + i)} + \frac{c}{s - (-1 - i)},
\]

then solve for the constants \( a, b, c \) and inverse transform to get

\[
a + be^{-t}e^{it} + ce^{-t}e^{-it},
\]

and finally take real parts (the coefficients \( b, c \) will be complex so this requires a bit of work).

### 2.3 Anticipating the answer

(This suggestion applies mostly to homework/exams). When taking the inverse transform to get \( y(t) \) from \( Y(s) \) when \( y(t) \) solves an ODE

\[
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = g(t),
\]

you already know what terms will appear in the homogeneous part of the equation. It may help to keep that in mind. For example, suppose you are tasked with using \( L \)

\[
y^{(4)} - 2y''' + 5y'' - 8y + 4 = 0, \quad y(0) = a, \ y'(0) = b, \ y''(0) = c, \ y'''(0) = d
\]

has characteristic polynomial \((\lambda - 1)^2(\lambda^2 + 4)\). The solution \( y(t) \) will be a linear combination of basis solutions are \( e^t, te^t, \sin 2t, \cos 2t \), which have transforms

\[
\frac{1}{s - 1}, \quad \frac{1}{(s - 1)^2}, \quad \frac{2}{s^2 + 4}, \quad \frac{s}{s^2 + 4}.
\]

So when taking the inverse transform of

\[
Y(s) = \frac{\cdots}{(s - 1)^2(s^2 + 4)}
\]

you know in advance what form to use in the partial fraction decomposition (linear combination of the four transforms listed above). That this works is an indication that solving for the homogeneous part of \( y(t) \) using \( L \) is more or less just an exercise (if you already know the basis solutions, the only practical reason to transform and then inverse transform is to practice applying \( L \)).

### 2.4 Derivatives

The \( t \)-derivative formula

\[
L[f'(t)] = sF(s) - f(0)
\]

(7)
can be used to compute inverse transforms, but do not forget the last term! For example, suppose we only know \( L[\sin t] = 1/(s^2 + 1) \) and want to compute \( L^{-1} \) of

\[
G(s) = \frac{s}{s^2 + 1}.
\]
Write this as \( G(s) = sF(s) - f(0) \) where \( F = 1/(s^2 + 1) \) and \( f = \mathcal{L}^{-1}[F] = \sin t \), so
\[
\mathcal{L}^{-1}[G(s)] = f'(t) = \cos t.
\]

Since \( f(0) = 0 \) the extra term did not matter. The same does not quite work for
\[
G(s) = \frac{s^2}{s^2 + 1}.
\]

Write \( G(s) = sF(s) \) where now \( F(s) = s/(s^2 + 1) \) and \( f(t) = \cos t \). To apply the formula (7) we need to pay attention to the fact that \( f(0) = 1 \):
\[
\mathcal{L}^{-1}[G(s)] = \mathcal{L}^{-1}[sF(s) - f(0)] + \mathcal{L}^{-1}[1]
= f'(t) + \mathcal{L}^{-1}[1]
= -\sin t + \delta(t).
\]

Of course one could also have just written \( G(s) = 1 - 1/(s^2 + 1) \) in the first place and obtained the same answer.

### 2.5 Using the convolution theorem [also in last week’s notes]

The convolution theorem is obviously useful for computing the transform of a convolution. It is also useful for computing inverse transforms of products (sometimes). Some examples:

<table>
<thead>
<tr>
<th>( \mathcal{L}^{-1} ) using the convolution theorem:</th>
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<tbody>
<tr>
<td><strong>Example 1:</strong> Let’s find the inverse transform of</td>
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<tr>
<td>[ F(s) = \frac{1}{s^3(s^2 + 1)}. ]</td>
</tr>
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We write \( F \) as a product of easier-to-invert functions and then inverse transform it using the convolution theorem. Let \( G(s) = 1/s^3 \) and \( H(s) = 1/(s^2 + 1) \) and let \( f = \mathcal{L}^{-1}[G], h = \mathcal{L}^{-1}[H] \). Then
\[
\mathcal{L}^{-1}[F(s)] = (f * g)(t).
\]

Both \( G \) and \( H \) are standard transforms; we find that \( g(t) = t^2/2 \) and \( h(t) = \sin t \). From here, either declare victory or evaluate the convolution integral explicitly (using some
integration by parts):

\[ \mathcal{L}^{-1}[F(s)] = \int_0^t f(t - \tau)g(\tau) \, d\tau \]

\[ = \int_0^t \frac{1}{2}(t - \tau)^2 \sin(\tau) \, d\tau \]

\[ = -\frac{1}{2}(t - \tau)^2 \cos \tau \bigg|_0^t + \int_0^t (t - \tau) \cos(\tau) \, d\tau \]

\[ = -\frac{1}{2}t^2 + (t - \tau) \sin(\tau) \bigg|_0^t - \int_0^t \sin(\tau) \, d\tau \]

\[ = -\frac{1}{2}t^2 + \cos t - 1. \]

Note that this approach is about the same amount of work as using partial fractions.

**Example 2:** It may be tempting to try to find \( \mathcal{L}^{-1} \) of

\[ F(s) = \frac{s}{s^2 + 1} \]

using the convolution theorem, by setting \( G(s) = s \) and \( H(s) = 1/(s^2 + 1) \). Then

\[ \mathcal{L}^{-1}[F(s)] = \int_0^t g(t - s) \sin(\tau) \, d\tau \]

where \( g(t) = \mathcal{L}^{-1}[G(s)] \). The problem is that \( \mathcal{L}[\delta'] = s\mathcal{L}[\delta] = s \) so \( g \) is not a function. One can get the right answer by substituting in \( g = \delta' \) but that requires some care (and technical issues!).

The easier approach is either to just recognize this is \( \cos t \) or to use partial fractions to write it as \( a/(s - i) + b/(s + i) \).