Topics covered

• Step functions
  ○ Solving IVPs with discontinuous forcing functions

• Convolutions
  ○ Definition/properties
  ○ Convolution theorem
  ○ Transfer function, Laplace vs. time domain solutions

• Impulse
  ○ Dirac delta ‘function’
  ○ Laplace transform of \( \delta(t) \)

• (Optional: proof of the convolution theorem)
1 Step functions

Suppose the output voltage \( y(t) \) of a circuit is related to the input voltage \( g(t) \) by

\[
y'' + by' + cy = g(t).
\]

We may want to determine what happens when a switch is suddenly turned on, then turned off after some time. A unit voltage applied from \( t = 1 \) to \( t = 2 \), for example, would correspond to the forcing function

\[
g(t) = \begin{cases} 
0 & t < 1 \\
1 & 1 < t < 2 \\
0 & t > 2 
\end{cases}.
\]

This function is piecewise continuous with jumps at \( t = 1 \) and \( t = 2 \). We know already that the Laplace transform is compatible with discontinuities; now we need to find a good way to represent such functions and to calculate the transforms.

1.1 Step functions: definitions

The unit step function or Heaviside function \( H(t) \) is the function that ‘jumps’ from zero to one at \( t = 0 \):

\[
u(t) = \begin{cases} 
0 & t < 0 \\
1 & t \geq 0.
\end{cases}
\]

Notation: By convention, \( u(0) \) is usually defined to be \( 1/2 \), but the value is not relevant to the discussion so we take \( u(0) = 1 \) for simplicity. Sometimes \( H(t) \) or even \( \theta(t) \) are used instead of \( u \).

The function that jumps from 0 to 1 at a point \( c \in \mathbb{R} \) can be written in terms of \( u \) by translation. It is convenient to define

\[
u_c(t) = u(t - c) = \begin{cases} 
0 & t < c \\
1 & t \geq c.
\end{cases}
\]

The function \( u_c(t) \) can be used as a way to turn on/off a term in a function, allowing us to write any piecewise continuous function as a sum of continuous functions times step functions. The reason for using this form is that the step function behaves nicely when transformed.

Basic approach: Go from left to right and add \((\cdots)u_c(t)\) at each discontinuity to account for the jump. For instance, given the strange function

\[
g(t) = \begin{cases} 
t^2 & t < 1 \\
4 & 1 < t < 2 \\
-5 & 2 < t < 3 \\
et^t & t > 3
\end{cases}
\]

first take

\[
g(t) = t^2 + \cdots.
\]
Now add a term \((\cdots)u_1(t)\) to deal with the discontinuity:
\[
g(t) = t^2 + (4 - t^2)u_1(t)
\]
which makes \(g\) correct up to \(t = 2\). Continuing,
\[
g(t) = t^2 + (4 - t^2)u_1(t) + (-5 - 4)u_2(t) + \cdots
\]
to make \(g(t) = -5\) for \(2 < t < 3\), and finally
\[
g(t) = t^2 + (4 - t^2)u_1(t) - 9u_2(t) + (e^t + 5)u_3(t).
\]

**Alternate approach:** For each case, we construct a function that is 1 in that interval and zero otherwise. This is easy since
\[
u_a(t) - u_b(t) = \begin{cases} 
  1 & a < t < b \\
  0 & \text{otherwise}.
\end{cases}
\]  
(1)

Writing \(g(t)\) in terms of these functions, each term will be ‘independent’ of the others (only one term will be non-zero at a given time \(t\)):
\[
g(t) = t^2(1 - u_1(t)) + 4(u_1(t) - u_2(t)) - 5(u_2(t) - u_3(t)) + e^t u_3.
\]
The above can then be easily converted into a sum of step functions (which will be most convenient for applying \(L\)). Some other examples follow:

### Representing discontinuities with step functions:

**Example 1:** A function with a jump discontinuity:
\[
f(t) = \begin{cases} 
  \sin t & t < \pi \\
  1 & t \geq \pi
\end{cases}
\]
This can be written as
\[
f(t) = \sin t(1 - u_\pi(t)) + u_\pi(t)
\]
or as a sum of step functions
\[
f(t) = \sin t + (1 - \sin t)u_\pi(t).
\]

**Example 2:** Consider a unit pulse starting at \(t = 1\) and ending at \(t = 2\):
\[
f(t) = \begin{cases} 
  0 & t < 1 \\
  1 & 1 < t < 2 \\
  0 & t > 2
\end{cases}
\]
Using the first approach, we write this in terms of step functions by taking
\[
f(t) = u_1(t) + \cdots
\]
which creates the jump from 0 to 1 at \( t = 1 \). Then we add a jump down by 1 at \( t = 2 \):

\[
f(t) = u_1(t) - u_2(t).
\]

The result also follows directly from the other approach (1).

**Example 3:** Even if there are infinitely many discontinuities, we can write the function in terms of steps. For instance,

\[
f(t) = \begin{cases} 
1 & 2n < t < 2n + 1 \\
-1 & 2n + 1 < t < 2n + 2 
\end{cases}, \quad t \geq 0.
\]

is a ‘square wave’, which looks like \( \sin \pi x \) but with discontinuous jumps from \(-1\) to \(1\). In terms of step functions, \( f(t) \) is given by

\[
f(t) = u_0(t) - 2u_1(t) + 2u_2(t) - 2u_3(t) \cdots = u_0(t) + 2 \sum_{k=1}^{\infty} (-1)^k u_k(t).
\]

Note that at a given value of \( t \), the sum has finitely many non-zero terms (why?) so the sum converges at every \( t \).

1.2 Transform of \( u_c(t) \)

The Laplace transform of the step function \( u_c(t) \) for \( c > 0 \) is

\[
\mathcal{L}[u_c(t)] = \int_0^\infty e^{-st} u_c(t) \, dt = \int_c^\infty e^{-st} \, dt = \frac{e^{-cs}}{s}, \quad s > 0.
\]

If \( c < 0 \) then \( \mathcal{L} \) does not ‘see’ the discontinuity (because then \( u_c = 1 \) for \( t > 0 \)).

The step function ‘cuts off’ the integral below \( t < c \) and leaves the rest. More generally,
consider the transform of a function \( u_c(t)f(t - c) \):

\[
\mathcal{L}[u_c(t)f(t - c)] = \int_0^\infty e^{-st}u_c(t)f(t - c) \, dt
\]

\[
= \int_c^\infty e^{-st}f(t - c) \, dt
\]

\[
= \int_0^\infty e^{-s(\tau + c)}f(\tau) \, d\tau \quad \text{(setting } \tau = t - c)\]

\[
= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau) \, d\tau
\]

\[
= e^{-cs}\mathcal{L}[f(t)].
\]

Again, it is important to emphasize that the formula holds only for \( c > 0 \).

### Transforms with step functions

**Example 1:** Let

\[
f(t) = \begin{cases} 
0 & t < \pi \\
sin(t - \pi) & \pi < t < 2\pi \\
0 & t > 2\pi
\end{cases}
\]

Then \( f(t) = \sin(t - \pi)u_\pi - \sin(t - \pi)u_{2\pi} \). Compute term by term:

\[
\mathcal{L}[\sin(t - \pi)u_\pi(t)] = e^{-\pi s}\mathcal{L}[\sin t] = \frac{e^{-\pi s}}{1 + s^2}.
\]

\[
\mathcal{L}[\sin(t - \pi)u_{2\pi}] = -\mathcal{L}[\sin(t - 2\pi)u_{2\pi}] = -\frac{e^{-2\pi s}}{1 + s^2}.
\]

Note that the function multiplying \( u_{2\pi} \) was rewritten in terms of \( (t - 2\pi) \) in order to apply the translation rule. The transform of \( f(t) \) is

\[
\mathcal{L}[f(t)] = \frac{e^{-\pi s} + e^{-2\pi s}}{1 + s^2}.
\]

**Example 2:** Let \( f(t) = e^{t-2} \) if \( t < 2 \) and \( f(t) = 1 \) if \( t > 2 \). Then

\[
f(t) = e^{t-2} + (1 - e^{t-2})u_2(t).
\]

We get

\[
\mathcal{L}[f(t)] = e^{-2}\mathcal{L}[e^t] + \mathcal{L}[(1 - e^{t-2})u_2(t)] = e^{-2}\mathcal{L}[e^t] + e^{-2s}\mathcal{L}[1 - e^t].
\]

Now apply the formulas \( \mathcal{L}[e^t] = 1/(s - 1) \) and \( \mathcal{L}[1] = 1/s \) to obtain

\[
\mathcal{L}[f(t)] = \frac{e^{-2}}{s - 1} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s - 1}\right).
\]
Thus we have yet another equivalence between functions and transforms:

\[ \text{‘translation’ right by } c \quad \iff \quad \text{multiplication by } e^{-cs}. \]

*Note: translation here means setting \( f \) to zero for \( t < c \). Through this rule, it is straightforward to transform functions with jump discontinuities and to solve IVPs with such functions (see example in the next section).

There is a dual correspondence with translation in the \( s \)-domain:

\[ \mathcal{L}[e^{ct}f(t)] = F(s - c), \quad \text{for } c \in \mathbb{R} \]

(check this!). The above is defined when \( s > a + c \) if \( |f(t)| \leq Ke^{at} \). Thus

\[ \text{multiplication by } e^{ct} \iff \text{translation by } c. \]

Note that here \( c \) is allowed to be negative. This latter rule is useful for dealing with factors of \( e^{ct} \) when computing \( \mathcal{L} \) and translation when calculating \( \mathcal{L}^{-1} \).

2 IVPs with discontinuous forcing

Here we solve an IVP with discontinuous forcing via the Laplace transform. The process is the same as for the IVPs solved earlier, except that we now need to use the rule (??) both in taking the transform and the inverse transform. The process typically goes as follows:

- Transform the ODE, using the transform formula for step functions,
- End up with \( Y(s) \) as a sum of terms like \( F(s)e^{-cs} \),
- Inverse transform using the formula again.

As an example, consider a mass attached to a damped spring which has displacement \( y(t) \) satisfying

\[ y'' + 2y' + 2y = g(t), \quad y(0) = y'(0) = 0. \]

The spring is initially at rest, then pulled with force 1 from \( t = 1 \) to \( t = 2 \) and let go, i.e.

\[ g(t) = u_1(t) - u_2(t). \]

The Laplace transform of \( g \) above is

\[ G(s) = \mathcal{L}[u_1(t)] - \mathcal{L}[u_2(t)] = e^{-s}/s - e^{-2s}/s. \]

Taking the Laplace transform of the IVP, we then find that

\[ Y(s) = (e^{-s} - e^{-2s})F(s), \quad F(s) = \frac{1}{s(s^2 + 2s + 2)}. \]
The inverse transform $f(t) = L^{-1}[F(s)]$ is obtained by using partial fractions. Write

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{b + c(s + 1)}{(s + 1)^2 + 1}.$$ 

Solving for $a, b, c$ and then using the general rule $L^{-1}[G(s - c)] = e^{ct}L^{-1}[G]$ and the standard transforms $L[\cos t] = s/(s^2 + 1)$ and $L[\sin t] = 1/(s^2 + 1)$ we get

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t.$$ 

The solution is therefore

$$y(t_0) = u_1(t)f(t - 1) - u_2(t)f(t - 2).$$ 

For $t > 2$ the solution is

$$y(t) = e^{-(t-1)}(\sin(t - 1) + \cos(t - 1) - e\sin(t - 2) - e\sin(t - 2)).$$ 

The damping causes the displacement to decay exponentially (while it oscillates).
3 The transfer function

Now we move on to develop more theory and some crucial aspects of the Laplace transform. To motivate the discussion that follows, let us view the ODE

\[ y'' + by' + cy = g(t) \]

as describing the response \( y(t) \) of a physical system to an input \( g(t) \). The two classical examples are the displacement \( y(t) \) of a spring with an applied force \( g(t) \), and the output of a circuit (e.g. an LRC circuit) with some input voltage/current.

We are interested in understanding how the output \( y(t) \) is related to the input \( g(t) \). To simplify matters, let's look at the system starting from rest, i.e.

\[ y'' + by' + cy = g(t), \quad y(0) = y'(0) = 0. \]  \hfill (2)

(The solution to this IVP in engineering is sometimes called the zero state response). Taking the Laplace Transform, we find (by the methods of last week) that there is a function \( H(s) \) independent of \( g(t) \) such that

\[ Y(s) = H(s)G(s) \]

where \( Y(s) = \mathcal{L}[y(t)], G(s) = \mathcal{L}[g(t)] \) (check this!). The function \( H(s) \) is called the transfer function, a key object in analyzing circuits and mechanical systems. For example, the ODE

\[ y'' + y = g(t) \]

has a transfer function \( H(s) = 1/(s^2 + 1) \).

The transfer function is the ratio of output to input:

\[ H(s) = \frac{Y(s)}{G(s)} \]

and is an inherent property of the system. Thus even though \( y(t) \) and \( g(t) \) might be related in a complicated way, the relation is simple in the Laplace domain (s-domain). As a diagram:

![Diagram](image)

The system takes the input at a value of \( s \) and scales it by \( H(s) \). The simplicity of the description makes the s-domain a good setting for analysis of such systems - e.g. we can figure out how to dampen unwanted frequencies of a signal by choosing \( H(s) \) to be small at
the unwanted values of \( s \) (low/high/band-pass filters). The details, of course, are left to an engineering course.

However, we are left with an important question - how are \( y(t) \) and \( g(t) \) related? You may recall that using variation of parameters we found examples where the solution to (2) was

\[
y(t) = \int_0^t h(t - s)g(s) \, ds
\]

where \( h \) is some function. For instance, \( y'' + y = g(t) \) had a particular solution

\[
y(t) = \int_0^t \sin(t - s)g(s) \, ds.
\]

Note that \( y(s) \) is certainly not a factor times \( g(s) \)! - it is more complicated. To understand the relationship, we need to define a new mathematical operation.

4 Convolutions

The convolution of two functions \( f(t) \) and \( g(t) \) defined on \([0, \infty)\) is

\[
(f * g)(t) = \int_0^t f(t - \tau)g(\tau) \, d\tau.
\] (3)

Aside on definition: We use the definition above because the Laplace transform is applied to functions on \([0, \infty)\). In some contexts, we instead have \( f \) and \( g \) defined on \((-\infty, \infty)\) and then the convolution is defined to be

\[
(f * g)(t) = \int_{-\infty}^\infty f(t - \tau)g(\tau) \, d\tau.
\]

If \( f(t), g(t) \) are only defined on \([0, \infty)\), we can extend them to be zero for \( t < 0 \) and then the two definitions are equivalent. For the purposes of this section, (3) is the most convenient definition except for one small technicality to be noted later.

4.1 Properties

The convolution obeys some product-like properties:

Commutativity: \( f * g = g * f \)

Associativity: \((f * g) * h = f * (g * h)\)

Distributivity: \( f * (g + h) = f * g + f * h \)

Otherwise, it behaves quite differently from the usual product:

- The value of \((f * g)(t)\) depends not just on \( f(t) \) and \( g(t) \) but on all values of the functions in \([0, t]\) (much more complicated than a product!).
• $f \ast 1 \neq f$; there is an ‘identity’ $g$ such that $f \ast g = f$ but it is not easy to define (we’ll see this in looking at impulse functions).

• ...Many other properies we won’t need here.

**Examples of convolutions:** If $f(t) = \sin t$ then

$$f \ast 1 = \int_0^t \sin(t - \tau) \, ds = \cos(t - \tau) \bigg|_0^t = 1 - \cos t.$$  

Note that $f \ast 1$ here denotes $f \ast g$ where $g$ is the constant function $g(t) = 1$. Sometimes it is easier to swap arguments so that the nicer function has the $t - \tau$ argument:

$$1 \ast f = \int_0^t \sin(\tau) \, d\tau = -\cos(\tau) \bigg|_0^t = 1 - \cos t.$$  

If $f(t) = e^t$ then

$$f \ast f = \int_0^t e^{t-s} e^s \, ds = \int_0^t e^t \, ds = te^t.$$  

### 4.2 (optional:) Visualization

It is sometimes useful to give a geometric interpretation of the convolution. We can do so in the following way. Observe that given a value of $t$, the value of the convolution

$$(f \ast g)(t) = \int_0^t f(t - \tau)g(\tau) \, d\tau$$

is the (signed) area under the curve $f(t - \tau)g(\tau)$. This curve is obtained by taking $f(-\tau)$, then shifting it right by $t$ and multiplying by $g(\tau)$. Thus we can:

• Extend $f, g$ to be zero for $\tau < 0$.

• Plot $g(-\tau)$ and $f(\tau)$.

• Imagine the graph of $g(-\tau)$ sliding to the right as $t$ increases.

• The area under the product of the two plotted curves is $(f \ast g)(t)$.

This is a sense in which the convolution is a kind of moving weighted average of two functions (in fact, in can be used directly for smoothing of data). As an example of the visualization, consider $g(\tau) = u_1(\tau) - u_2(\tau)$ (a box of height 1 on $[1, 2]$) and $h(t) = \sin \pi \tau$. Diagrams not included here; see MATLAB code posted elsewhere.
4.3 Laplace transform

The Laplace transform of a convolution is important enough to have its own theorem\(^1\):

**Convolution theorem (for Laplace transform):** Suppose \( F(s) = \mathcal{L}[f(t)] \) and \( G(s) = \mathcal{L}[g(t)] \) are defined for \( s > a \). Then

\[
\mathcal{L}[(f * g)(t)] = F(s)G(s),
\]

This establishes a key correspondence: *convolutions* in the \( t \)-domain correspond to *products* in the \( s \)-domain. This allows convolutions to be handled via transforms (useful in many problems where convolutions appear), and also lets us take inverse transforms of products.

Immediately, it answers the question of how to relate \( y(t) \) to \( g(t) \) in the ODE (2). For

\[
y'' + by' + cy = g(t), \quad y(0) = y'(0) = 0
\]

we know the transform of the solution is

\[
Y(s) = H(s)G(s).
\]

Now let \( h(t) = \mathcal{L}^{-1}[H(s)] \). Then by the convolution theorem,

\[
y(t) = \int_0^t h(t-s)g(s) \, ds = (h * g)(t).
\]

Thus in the \( t \)-domain, the output \( y(t) \) and input \( g(t) \) to the system are related by a convolution of \( g \) with a function \( h(t) \) depending on the ODE.

4.4 Non-zero initial conditions

Suppose now that \( y(0) = y_0 \) and \( y'(0) = y'_0 \). Then

\[
Y(s) = H(s)(G(s) + sy_0 + y'_0)
\]

In this case we obtain the solution in terms of particular/homogeneous solutions as follows:

\[
y(t) = \underbrace{(h * g)(t)}_{y_p(t)} + \underbrace{\mathcal{L}^{-1}[H(s)(sy_0 + y'_0)]}_{y_h(t)}.
\]

One might be tempted to evaluate the second term as a convolution as well; but that would require taking \( \mathcal{L}^{-1}[1] \) and \( \mathcal{L}^{-1}[s] \). **It is important to note that** with what we know so far, the formula

\[
\mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]
\]

only applies when \( F \) and \( G \) are the Laplace transforms of functions \( f \) and \( g \). Unfortunately, there is no function \( f(t) \) such that

\[
\mathcal{L}[f] = 1.
\]

To get around the issue, we will need to do more work - the subject of the next section.

\(^1\)For the interested reader, a proof is at the end of the notes.
4.5 Using the convolution theorem

The convolution theorem is obviously useful for computing the transform of a convolution. It is also useful for computing inverse transforms of products (sometimes). Some examples:

$L^{-1}$ using the convolution theorem:

**Example 1:** Let’s find the inverse transform of

$$F(s) = \frac{1}{s^3(s^2 + 1)}.$$  

We write $F$ as a product of easier-to-invert functions and then inverse transform it using the convolution theorem. Let $G(s) = 1/s^3$ and $H(s) = 1/(s^2 + 1)$ and let $f = \mathcal{L}^{-1}[G], h = \mathcal{L}^{-1}[H]$. Then

$$\mathcal{L}^{-1}[F(s)] = (f * g)(t).$$

Both $G$ and $H$ are standard transforms; we find that $g(t) = t^2/2$ and $h(t) = \sin t$. From here, either declare victory or evaluate the convolution integral explicitly (using some integration by parts):

$$\mathcal{L}^{-1}[F(s)] = \int_0^t f(t - \tau)g(\tau)\,d\tau$$

$$= \int_0^t \frac{1}{2} (t - \tau)^2 \sin(\tau)\,d\tau$$

$$= \left[-\frac{1}{2} (t - \tau)^2 \cos \tau\right]_0^t + \int_0^t (t - \tau) \cos(\tau)\,d\tau$$

$$= \left[-\frac{1}{2} t^2 + (t - \tau) \sin(\tau)\right]_0^t - \int_0^t \sin(\tau)\,d\tau$$

$$= -\frac{1}{2} t^2 + \cos t - 1.$$

Note that this approach is about the same amount of work as using partial fractions.

**Example 2:** It may be tempting to try to find $\mathcal{L}^{-1}$ of

$$F(s) = \frac{s}{s^2 + 1}$$

using the convolution theorem, by setting $G(s) = s$ and $H(s) = 1/(s^2 + 1)$. Then

$$\mathcal{L}^{-1}[F(s)] = \int_0^t g(t - s) \sin(\tau)\,d\tau$$

where $g(t) = \mathcal{L}^{-1}[G(s)]$. The problem is that $\mathcal{L}[\delta'] = s\mathcal{L}[\delta] = s$ so $g$ is not a function. One can get the right answer by substituting in $g = \delta'$ but that requires some care (and technical issues!).

The easier approach is either to just recognize this is $\cos t$ or to use partial fractions to write it as $a/(s - i) + b/(s + i)$. 

12
5 Delta functions

In this section we will see a variety of ways to motivate the same thing. To start, let’s forget about convolutions and the Laplace transform for a little bit and consider a new problem. Let’s consider an ODE describing, say, a mass spring system where \( g(t) \) is the applied force.

Main question: How do we represent a unit impulse, i.e. an impulse of 1 applied to a system instantaneously at \( t = 0 \)?

Technical note: Here we will also forget about the \( t \in [0, \infty) \) restriction used for the Laplace transform in the past and allow negative times; it will be addressed later.

There is no trouble with representing a unit impulse applied over an interval, so we can try to do that and shrink the interval to zero. With this in mind, we define a sequence of forcing functions \( g_n(t) \) that impart a unit impulse over smaller and smaller intervals as \( n \to \infty \):

\[
g_n(t) = \begin{cases} 
\frac{n}{2} & -\frac{1}{n} < t < \frac{1}{n} \\
0 & \text{otherwise}
\end{cases}
\]

**Notation:** You will often instead see \( g_\epsilon(t) = \frac{1}{2\epsilon} \) for \( |t| < \epsilon \), defined for all \( \epsilon > 0 \). Limits are then taken as \( \epsilon \to 0 \). This approach is equivalent to the sequence version above; the theory is not actually that sensitive to the way the ‘approximating’ impulse functions \( g_\epsilon \) are defined.

The key property is that

\[
\int g_n(t) \, dt = 1.
\]

We’d like to take a ‘limit’ as \( n \to \infty \) to obtain a function \( g(t) \) that has a total impulse of 1 but is zero away from zero, i.e.

\[
\int_{-\infty}^{\infty} g(t) \, dt = 1, \quad g(t) = 0 \text{ for } t \neq 0.
\]

The limit of the impulse is not a problem:

\[
\lim_{n \to \infty} \int g_n(t) \, dt = \lim_{n \to \infty} 1 = 1,
\]
but the (pointwise) limit of the functions is nonsense:

\[
\lim_{n \to \infty} g_n(t) = \begin{cases} 
0 & t \neq 0 \\
\infty & t = 0.
\end{cases}
\]

The expression above is of no use, except to show that we need some other notion of ‘limit’ and ‘function’ to describe what we want.

Observe that other limits can be taken when the \(g_n\)’s are in an integral sign. In particular,

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(t) f(t) \, dt = f(0).
\]

The proof can be found in the text. Intuitively, view this as follows: as \(n \to \infty\), the interval where \(g \neq 0\) shrinks to a smaller and smaller region around zero. Thus the contribution to the integral must come only from around \(t = 0\). In this region, \(f(t) \approx f(0)\) so

\[
\int g_n(t) f(t) \, dt \approx f(0) \int g_n(t) \, dt \to f(0) \quad \text{as} \quad n \to \infty.
\]

5.1 The Dirac delta

The idea is now to define an object \(\delta(t)\), called the Dirac delta function (confusingly, it is not a function!) that obeys the following rules:

\[
\begin{align*}
(i) \quad & \int_{-\infty}^{\infty} \delta(t) \, dt = 1, \\
(ii) \quad & \delta(t) = 0 \quad \text{for}\ t \neq 0, \\
(iii) \quad & \int_{-\infty}^{\infty} \delta(t-a) f(t) \, dt = f(a) \quad \text{for all continuous functions} \ f(t).
\end{align*}
\]

The object \(\delta\) under an integral sign can be thought of as convenient notation for the limit from the previous section, e.g.

\[
\int_{-\infty}^{\infty} \delta(t) \, dt \quad \text{is shorthand for} \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(t) \, dt.
\]

We can develop a series of rules for manipulating \(\delta\) and not have to worry about what it actually means (evaluating the \(\epsilon \to 0\) limits). Again, to emphasize, despite writing \(\delta(t)\) etc. the Dirac delta is not a function; it is notation\(^{\text{2}}\).

The Dirac delta has some function-like properties when it is under an integral sign. For example, one can translate the delta function, which lets one write things like, e.g.

\[
\int_{-\infty}^{\infty} \delta(t-\tau) f(\tau) \, d\tau = \int_{-\infty}^{\infty} \delta(\tau) f(t-\tau) \, d\tau = f(t)
\]

\(^{2}\text{One can define }\delta\text{ more rigorously but it requires some subtle and advanced mathematics that are not important to practical use. It is worth noting that }\delta\text{ was used to solve problems long before the rigorous theory was developed.}\)
using property (iii). Translating \( \delta \) moves the point at which the mass/impulse is concentrated (i.e. \( \delta(t-t_0) \) is zero away from \( t_0 \)). The above shows that the convolution of \( \delta \) and a function \( f \) can be computed and
\[
\delta \ast f = f \ast \delta = f.
\]
Thus \( \delta \) is an analogue of a multiplicative identity for convolutions. Similarly, if \( c > 0 \) then
\[
\mathcal{L}[\delta(t-c)] = \int_0^\infty e^{-st}\delta(t-c) \, dt = e^{-cs},
\]
which establishes (formally!) the Laplace transform of \( \delta \). One should take on faith that \( \delta \) is well-defined enough that the above makes sense. We then take \( c \to 0 \) to obtain (with some hand-waving!)
\[
\mathcal{L}[\delta] = 1.
\]
The above is the Laplace transform of \( \delta \). It also gives us the formula
\[
\mathcal{L}^{-1}[1] = \delta.
\]
so we can now inverse transform constant functions.

\[\text{optional: A technical issue:}\] There is a small technicality, however: If we try to compute \( \mathcal{L}[\delta] \) using the limit rule:
\[
\mathcal{L}[\delta] = \int_0^\infty e^{-st}\delta(t) \, dt = \lim_{n \to \infty} \int_0^\infty e^{-st}g_n(t) \, dt
\]
then the integral will miss half of the mass of \( \delta(t) \) (since half of the mass of \( g_n(t) \) is in \([0, 1/n]\) and the other half is in \([-1/n, 0]\)), leading to \( \mathcal{L}[\delta] = 1/2 \) (wrong).

The issue is that the definition of the Laplace transform is not compatible with our ‘rules’ for \( \delta \) when applying \( \mathcal{L} \) to \( \delta(t) \). One can either add an extra caveat to the rules for \( \delta \) or define the Laplace transform to be
\[
\mathcal{L}[f(t)] = \lim_{a \to 0^-} \int_a^\infty e^{-st}f(t) \, dt.
\]
When \( f \) is a piecewise continuous function, the limit is irrelevant (just put \( a = 0 \) in the first place). But for \( \delta \) it matters: at any finite \( a < 0 \), we always pick up all of the mass of \( \delta \), so nothing is lost and \( \mathcal{L}[\delta] = 1 \) by the rules we just created instead by a ‘magic’ limit.

The reason for the hand-waving is that the formula for \( \mathcal{L} \) we have is valid for functions but not quite valid for objects like \( \delta \). In practice, the technicalities will (probably) never get you into trouble so it is safe to ignore them.

5.2 Transform of \( \delta \)

(Continued in next week’s notes).
(optional:) Proof of the convolution theorem

We restate the theorem precisely first:

**Convolution theorem for Laplace transform:** Let \( f(t) \) and \( g(t) \) be piecewise continuous functions and suppose \( F(s) = \mathcal{L}[f(t)] \) and \( G(s) = \mathcal{L}[g(t)] \) exist for \( s > a \). Then

\[
\mathcal{L}[(f \ast g)(t)] = F(s)G(s)
\]

where \( (f \ast g)(t) = \int_0^t f(t - \tau)g(\tau)\,d\tau. \)

The proof is, for the most part, just a manipulation of some integrals. First we get \( F(s)G(s) \) to be one function under a double integral by writing each part with different integration variables:

\[
F(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi, \quad G(s) = \int_0^\infty e^{-st} g(\tau)\,d\tau. \tag{4}
\]

We then have

\[
F(s)G(s) = \int_0^\infty \int_0^\infty e^{-s(\xi + \tau)} f(\xi)g(\tau)\,d\xi\,d\tau.
\]

Now replace the inner variable \( \xi \) with \( t = \xi + \tau \) (and leave \( \tau \)):

\[
= \int_0^\infty \int_\tau^\infty e^{-st} f(t - \tau)g(\tau)\,dt\,d\tau
\]

and swap the integration order to get

\[
= \int_0^\infty \int_0^t e^{-st} f(t - \tau)g(\tau)\,d\tau dt
\]

\[
= \int_0^\infty e^{-st} \left( \int_0^t f(t - \tau)g(\tau)\,d\tau \right) dt
\]

\[
= \mathcal{L}[(f \ast g)(t)].
\]

For the integration order, note that the region of integration in the \((t, \tau)\) plane is \( \{t > \tau\} \), so if \( t \) is the outer variable then \( t \) ranges from 0 to \( \infty \) and \( \tau \) from 0 to \( t \) (draw a picture for this).

**Remark:** The trick of writing the transforms with different integration variables as in (4) is a common technique in analysis for manipulating identities involving integral formulas - just be careful when swapping integration order!