Note: Problems P1 and 5.2.10 are much longer than the others! The amount of work involved is typical for finding series solutions to ‘realistic’ differential equations (i.e. not ones that are chosen to be nice examples).

Text/AHP Problems:
- Section 5.1: 5, 12, 17, 25
- Section 5.2: 10
- Section 5.3: 2, 8

Other problems.

P1 (Legendre Equation). This is problem 22/23 in 5.3, edited.

The Legendre equation is

\[(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,\]

where \(\alpha > -1\) is a constant. This equation arises often in physics (which we’ll see in Chapter 10) - for instance, in calculating the electric field in a charged sphere.

a) Verify that \(x = 0\) is an ordinary point.

b) Find two (linearly independent) series solutions \(y_1\) and \(y_2\) around \(x = 0\) of the form

\[y_1 = \sum_{m=0}^{\infty} a_{2m}x^{2m}, \quad y_2 = \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1}.
\]

For the coefficients, find recurrence relations in the form

\[a_{2m} = (\cdots)a_{2m-2}, \quad a_{2m+1} = (\cdots)a_{2m-1}\]

which you do not need to ‘solve’ explicitly (see 5.3.22 for the result).

c) Using the ratio test, find the radius of convergence for \(y_1\) and \(y_2\) when \(\alpha\) is not an integer.

d) Show that if \(\alpha = 2n\) for a non-negative integer \(n\) then \(y_1\) from (b) is a polynomial of degree \(2n\). Find these polynomials for \(\alpha = 0\) and \(\alpha = 2\).

e) Show that if \(\alpha = 2n + 1\) for a non-negative integer \(n\) then \(y_2\) from (b) is a polynomial of degree \(2n + 1\). Find these polynomials for \(\alpha = 1\) and \(\alpha = 3\).
P2 (Dividing by power series). Given two power series
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \]
we often want to compute the series for the ratio,
\[ \frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} c_n x^n. \]

One way to do this is to multiply by the denominator and instead solve
\[ \sum_{n=0}^{\infty} a_n x^n = \left( \sum_{n=0}^{\infty} c_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) \]
for \( c_n \) (multiply it out, then equating powers of \( x^n \) on LHS/RHS).

a) Find a formula for \( c_0, c_1 \) in terms of the coefficients \( a_n \) and \( b_n \).

b) Using this approach, find a recurrence relation for the coefficients in the power series for
\[ f(x) = \frac{1}{1 - x - x^2}. \]
Write out the terms up to \( x^5 \). What is the radius of convergence? \textit{Hint: The ratio test can be used but is not necessary.}